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## STUDENTS' GEOMETRIC THINKING ON TRIANGLES: MUCH IMPROVEMENT IS NEEDED

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### ABSTRACT

A look into students' misconceptions help explain the very low geometric thinking and may assist teachers in correcting errors to aid students in reaching a higher van Hiele geometric thinking level. In this study, students' geometric thinking was described using the van Hiele levels and misconceptions on triangles. Participants (N=30) were Grade 9 students in the Philippines. More than half of the participants were in the van Hiele's visualization level. Most students had imprecise use of terminologies. A few had misconceptions on class inclusion, especially when considering isosceles right triangles and obtuse triangles. Very few students correctly recognized the famous Pythagorean Theorem. Implications for more effective geometry teaching are considered.

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## 1. INTRODUCTION

Geometry is the branch of Mathematics that studies shapes and measurement. It has been a subject of practical application in surveying, navigation, architecture, and engineering, among others. However, scholars assert that there is more than the practical application of geometry that makes it a "must-know". It is also a training ground to sharpen reasoning and problem solving skills (Johnston-Wilder & Mason, 2005)

Students' difficulty in mastering geometric concepts, particularly in formal proving, has been a dilemma shared across the globe. Studies (Fitriyani et al., 2018; Fuys et al., 1988; Gutiérrez et al., 1991; Senk, 1989) have reported that a majority of students who finished a formal geometry class did not completely reach van Hiele's level 4 on formal deduction. The study by Senk (1989) revealed a large percentage of students finishing high school in the United States only acquired a van Hiele Level 1 or 2. A study in the Philippines, conducted by Contreras (2009), disclosed that many students fall under level 2 while some are in transition between levels 3 and 4.

Atebe (2008) classified students' misconceptions in geometry based on his review of literature of previously conducted studies. Some illustrations were provided to explain each type of misconception in geometry. Nonetheless, an in-depth closer look into students' misconceptions specifically focused on triangles can help shed light regarding why they have very poor geometric thinking and how these can be addressed to aid students in reaching higher van Hiele geometric thinking level. Knowledge of these misconceptions can help teachers become proactive to consider preventive measures or to be reactive and remedy these errors. Research shows that teachers who take into account students' prior knowledge in planning their lessons can better promote conceptual understanding (Banerjee & Subramaniam, 2012).

### 1.1. The van Hiele Theory of Geometric Thinking

The van Hiele Theory is an empirically tested theory of learning in geometry asserting that students pass through five hierarchical levels of geometric thought upon proper instruction. The van Hiele Theory also recommends a particular order of instruction that can enable students to progress to a consecutively higher level (see Table 1). It is grounded on the premise that a student undergoes five developmental sequences of geometric thinking (Mason, 1998; Sarama et al., 2011).

**Table 1.** The van Hiele levels of geometric thinking (van de Walle et al., 2019)

Level	Students can
1 – Visualization	Recognize shapes merely by their appearances.
2 – Analysis	Recognize shapes and figures by their parts but unable to explain the relationship among these.
3 – Informal Deduction	<ul style="list-style-type: none"> <li>- Comprehend relationship between and among geometric properties;</li> <li>- Give own formal definitions or concepts; and</li> <li>- give informal reasoning using “if-then” statements.</li> </ul>
4 – Formal Deduction	<ul style="list-style-type: none"> <li>- grasp geometry fully as a system;</li> <li>- can do a formal proof.</li> </ul>
5 – Rigor	<ul style="list-style-type: none"> <li>- understand the relationship between various geometry systems;</li> <li>- compare, analyze, and prove in different Geometry systems even in the absence of concrete objects.</li> </ul>

The van Hiele model has proven to be a valid framework to assess and describe students' progress of geometric understanding and for designing instructional activities that cater to that level (Jones, 2003; van de Walle et al., 2019). If a teacher knows at which van Hiele level the student is, the educator understands where the student is operating and should be heading to next (Lim, 2011).

## 1.2. Misconceptions

Crawford (2001) defined misconceptions as “conceptual or reasoning difficulties that hinder students’ mastery of any discipline” and Drews et al. (2005) described it as the result of “a misapplication of a rule, an over- or under-generalization, or an alternative conception of the situation.” Misconception can occur as a natural stage of conceptual development (Swan, 2001), but must be corrected to overcome difficulties in understanding concepts (van der Sandt & Nieuwoudt, 2003). On the other hand, not all errors are consequences of misconceptions as some of these may arise from carelessness, misinterpretations of symbols or text (Swan, 2001), or from making wrong assumptions (Confrey, 1990).

Misconceptions can be formatively assessed so teachers can design and deliver remedial instruction to correct them in time for the summative assessment (Atebe & Schafer, 2010). Effective teachers take these misconceptions as powerful learning opportunities (Luneta, 2015). They understand that their critical role is to anticipate these misconceptions in their lesson planning and to have an array of approaches at their disposal to address head-on, common misunderstandings before these misconceptions stay on, worsen, and undermine confidence (Bamberger et al., 2010). Atebe (2008) generated a summary of the different misconceptions held by students in triangles and quadrilaterals (see Table 2).

**Table 2.** Students’ misconceptions in geometry as classified by Atebe (2008)

Misconception	Description
Imprecise Terminology	Lack of proper vocabulary
Identification/classification of basic shapes	Failure to correctly identify the name of a shape (Mayberry, 1983)
Class inclusion	Inability to recognize the inclusion of shapes within a larger category; This impedes geometric progress on reasoning about relationships.
Parallelism and Perpendicularity	Failure to correctly identify angle relationships formed when parallel lines are cut by a transversal or any properties brought by perpendicular lines
Angle sum of a triangle	Failure to use this relevant theorem in finding the measure of the third angle given the measures of the other two angles
Properties of shapes	Inability to describe explicitly the properties of triangles which includes relating the sides, angles, and the type of triangle

The choice for the topic under study is triangles. French (2004) argues that triangles are the key building blocks of geometric configurations and are known for its feature of being most stable and rigid. Students’ poor understanding of the concepts of triangles, as a basic polygonal shape, consequently leads to poor performance of subsequent polygonal shapes such as the quadrilateral.

### 1.3. Statement of the Problem

This study aimed to describe the geometric thinking and misconceptions of students as they think about triangles. Particularly, it addresses the following questions: (1) What are the students' van Hiele levels of geometric understanding of the triangles? (2) What are students' misconceptions of triangles?

## 2. METHOD

Participants (N=30) consisted of Grade 9 students (average age=15 years old) from a regular public high school in the Philippines. An intact class of students was provided by the school principal after permission to conduct the study was granted. A written informed consent was secured from their parents to allow voluntary participation in the study. From the initial 35 participants who gave their parents' consent, 30 came for the actual test. All students were assigned numbers and thus remained anonymous. All students had already gone through formal instruction in geometry prior to this study. In the Philippines, students are taught two-dimensional basic shapes as early as Grade 1, measurements in Grade 3, lines, line segments, angles and quadrilaterals in Grade 4, sides and angles of polygons in Grade 7, the axiomatic structure of geometry, triangle congruence, parallel and perpendicular lines in Grade 8, and parallelograms and triangle similarities in Grade 9. Each participant went through the test one at a time for 20-30 minutes to allow observation and further probing of student responses for their justifications of their answers. They were given the liberty to use a language with which they were comfortable. All students reside in the metropolitan area of the Philippines, belong to lower income families and attend free public education, can speak Filipino as their mother tongue, and speak English as their second language. The Philippines during this study was using the Mother Tongue-Based Multilingual Education as its banner program with Filipino as the medium of instruction from Kindergarten to Grade 3, and then the use of both Filipino and English as the language of instruction after Grade 3 (Metila et al., 2016).

Since the study sought to describe students' geometric understanding of triangles in each van Hiele level, and their misconceptions in triangles were identified using students' verbatim response and proofs, the study employed a descriptive research design. There was no intervention introduced and the study was conducted in the respondent's natural environment. The scope did not go beyond the formal deduction level of van Hiele since the highest level which is rigor, by theory, requires Non-Euclidean Geometry systems and these are not included in the high school curriculum.

The students' van Hiele level on geometric understanding was measured by a set of open-ended questions that probed students' conceptual understanding and reasoning skills that typify the level. Questions in Level 1 asked students to identify which of the given figures are triangles and which ones are right triangles. Level 2 questions asked students to apply the triangle angle sum theorem, to state the Pythagorean Theorem and identify the hypotenuse. Level 4 items asked students to prove two triangles congruent and similar. The van Hiele test was adapted from Senk (1989), Contreras (2009), Mayberry (1983) and de Villiers (2010) on triangles. It was validated by two content experts. Their comments were taken to revise and improve the items. A pilot-test was conducted with eight students who were pre-service teachers. This allowed the researchers to further refine the questions.

**Table 3.** Rubric in scoring students’ proofs adopted from Brandell (1994)

Number of Points	Experiments
5	The proof is correct as written.
4	The proof is correct for the most part, but it is missing a minor point; a “statement” may have an incorrect “reason”.
3	The proof is generally correct, but it is missing a few minor points or a major point.
2	The proof goes in a direction that is totally incorrect.
1	The proof restates the “Given” information but contains very little else.

The transcript of the interviews and solutions in the tests were analyzed by the first two researchers. In the event that scores disagreed, the third researcher broke the tie. Responses to each question in levels 1 to 3 were scored using a rubric. The rubrics were prepared by the researchers and approved by two Mathematics teachers who had more than 5 years of teaching experience. A different rubric adopted from Brandell (1994) was used to score students’ proofs in the fourth part of the van Hiele test (see Table 3).

The success criterion for each van Hiele level was based on Mayberry’s (1983), except in Levels 1 and 2. A student was considered to have attained the level if the score was at least the required percentage score in that van Hiele level. Success criteria in percent score in level 1, 2, 3, and 4 were 75, 70, 65 and 60, respectively, out of 100. Since the lower level items were easier than those in the higher levels, a greater percentage of the total points in the lower levels must be earned compared to the higher levels. The highest level attained by a student was considered the van Hiele level that he was able to achieve.

Students’ reasons were systematically and objectively characterized and compared against the correct reasoning. Misconceptions were classified based on Atebe’s (2008) list in Table 2 taking into consideration strict observance of the descriptions in each category.

### 3. RESULTS AND DISCUSSION

#### 3.1. Results

##### 3.1.1. Students’ van Hiele Levels

Table 4 summarizes students’ van Hiele test results per level in comparison to the success criteria score. Students’ mean score indicates that as a cohort, they failed to reach the success criteria score in all four levels. The predominantly attained level was visualization (level 1) as this had the greatest number of students who passed the required score. It can be seen that there are fewer students who passed a van Hiele level as we go from Level 1 through Level 4.

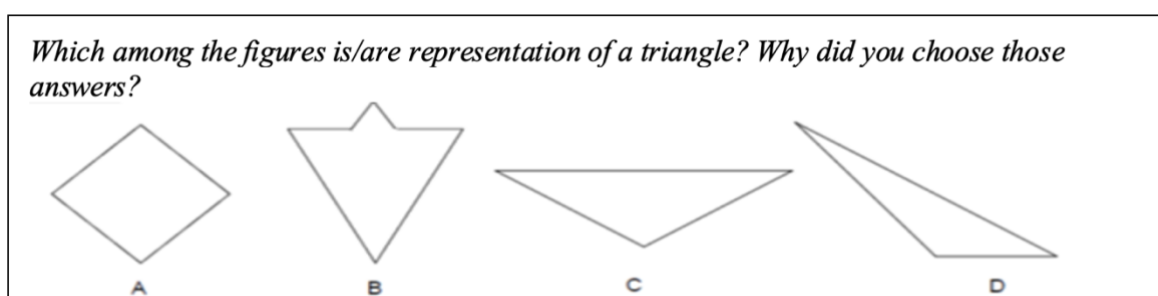
**Table 4.** Students' scores in the van Hiele test

	Level 1	Level 2	Level 3	Level 4
Success Criteria Score (Out of 100)	75	70	65	60
Students' Mean Score (Out of 100)	76.00	57.55	48.33	35
Number of Students who Passed (N=30)	20	12	9	7

The results revealed that, while at their high school, students were expected to have attained level 4 – Formal Deduction. Yet, only seven out of the 30 students reached this level. Most students (18) were only able to reach level 1 (Visualization). The majority of students reaching only level 1 is indicative that most of them could only recognize geometric shapes based on their appearance and not on their properties.

### *Level 1 – Visualization*

Twenty-six students were able to correctly identify which shapes are triangles (see [Figure 1](#)). But four of them failed to give the correct justification. For example, S29 said shapes C and D are triangles because shape C is an equilateral triangle and D is a representation of an isosceles triangle. Here, using “kind of triangle” as justification was uncalled for, thus, only three out of five points were given to this kind of answer.

**Figure 1.** Question 1.1 in the van Hiele test

In question 1.2, the correct answers are the shapes A, B, D, H and I (see [Figure 2](#)). Most of the students identified shapes H and D because of the  $90^\circ$  and the right-angle symbol, respectively. Twenty-three students answered B because they noticed that the sum of the two acute angles add up to  $90^\circ$ , leaving the third angle measuring  $90^\circ$  (see [Figure 2](#)).

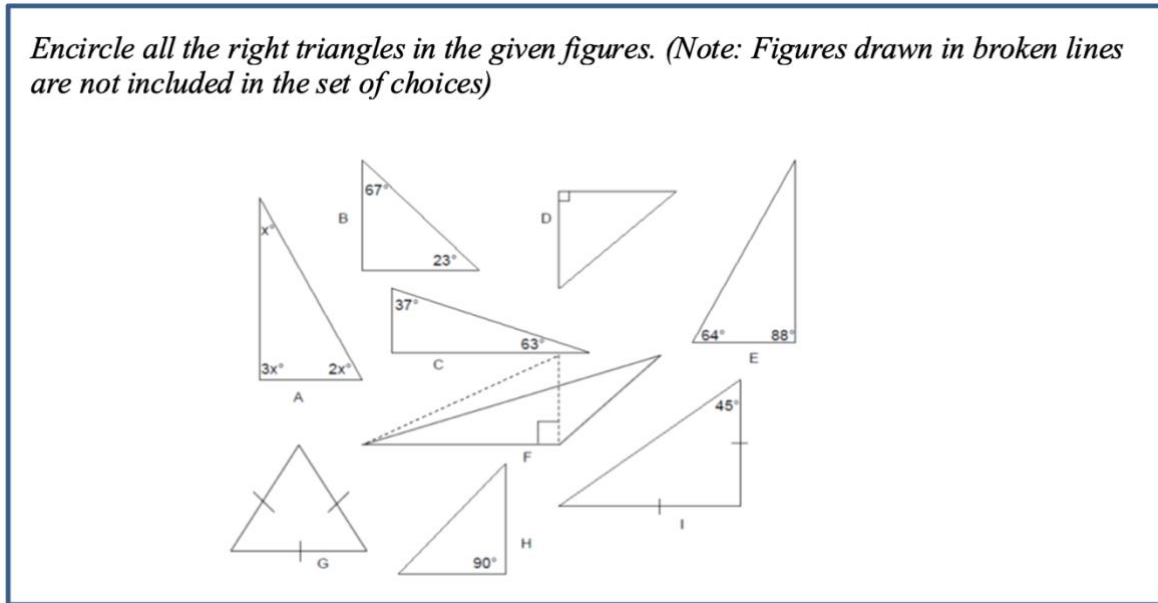


Figure 2. Question 1.2 in the van Hiele test

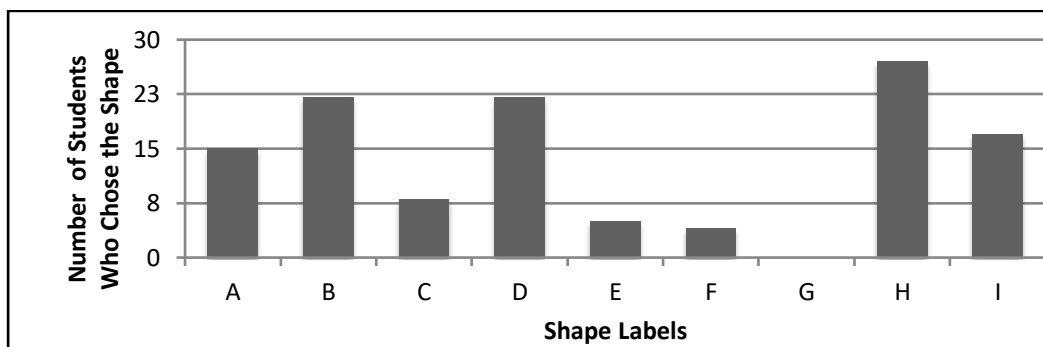


Figure 3. Students' responses to question 1.2

Seventeen students recognized that Shape I is a right triangle (see Figure 3), but only seven of them were able to explain the reason. An example of correct reasoning is shown in the following transcript.

S13 : *Kasi congruent 'tong side na 'to at ito. Ang measure nito ay 45 and 45 din ito kaya ang natitirang measure niya ay 90 degrees. (Because this side is congruent to this, [pointing to the sides with tick marks], its measure is 45 [referring to the angle with 45° label] and this is also 45 [pointing towards the other angle], so what's left with the third angle is 90 degrees.*

Half of the respondents did not notice Triangle A as another correct answer. Only three students were able to provide a correct argument. See an example below.

S19 : *Kinuha ko yung  $x + 2x + 3x$  is equal to 180 kasi yung sum ng interior angles of triangle is 180. So  $6x$  po. Divide both sides by 6,  $x$  is 30. Then 30 times 3, 90 sya so right triangle sya. Ayon po sa definition ng right triangle. (I got  $x + 2x + 3x$  equal to 180 because this is the sum of interior angles of a triangle. So  $6x$ . Then divide both*

sides by 6,  $x$  is 30. Then 30 times 3 is 90. So, it is a right triangle based on the definition of a right triangle.)

### Level 2 - Analysis

Question 2.1 asked for the measure of one of the base angles of an isosceles triangle given the measure of its vertex angle, which is  $140^\circ$ . Only 50% of the students gave the correct answer using the triangle angle sum theorem.

Question 2.2 asked for the relationship of the sides of a right triangle; whether or not a right triangle has a longest side, and if yes, which one. Unexpectedly, only three out of the 30 respondents recognized and stated the Pythagorean Theorem.

Thirteen respondents knew a right triangle has a longest side and correctly named it as the hypotenuse. The rest recognized the existence of a longest side, but failed to give its name.

### Level 3 – Informal Deduction

Students in this level were expected to recognize the relationship of shapes and their formal definitions.

Question 3.1 asked whether a right triangle can be isosceles. A majority of the students answered “yes” although some reasons were found to be inconsistent with their answer and a few of them did not give any reason. For instance, S25 said that this case is possible if one side is not congruent to any of the other sides. Other students answered “no” with the following reasons:

S17 : *Because in an isosceles triangle, you can't form a square (referring to the perpendicular symbol) while in a right triangle you can form a square inside.*

S18 : *Because an isosceles triangle has all equal parts or sides.*

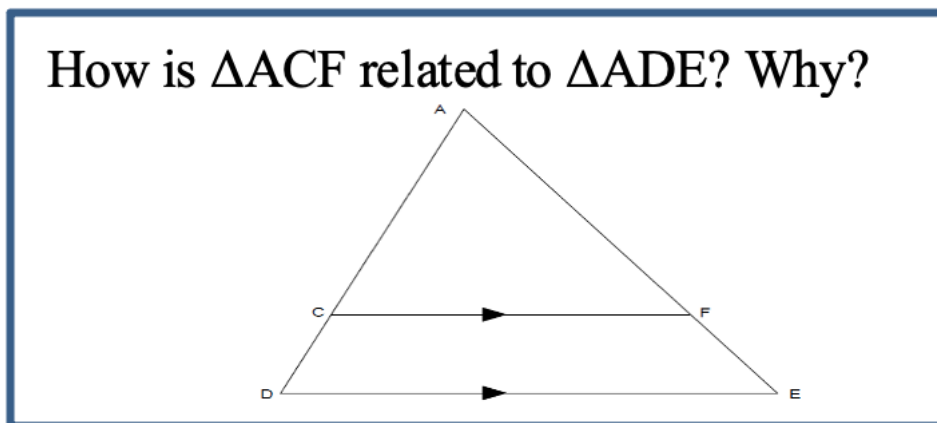
*Two angles and the included side of a triangle are congruent to two angles and the included side of another triangle. Then the two triangles are \_\_\_\_\_ (always, sometimes, or never) congruent to each other. Explain your answer.*

**Figure 4.** Question 3.2 in the van Hiele test

In Question 3.2 (see [Figure 4](#)), only one student (S29) said “sometimes” to the completion of the statement and explained “...because one of the other sides may be greater than one side of the other triangle.” The rest of the respondents correctly answered “always,” although some of their justifications were questionable as some admitted they cannot recall this postulate. Other students gave “Parallel Postulate,” “Isosceles Triangle Theorem,” and “SSS Postulate.”

In Question 3.3 (see [Figure 5](#)), only eight of the participants recognized that the relationship is similar, albeit all except one gave reasons that were questionable. Two of them simply stated the premise: “because the bases are parallel” and did not elaborate how this makes similar triangles. S01 was unable to identify relevant necessary and sufficient conditions to justify why the triangles are similar. Instead, he tried to describe the corresponding congruent angles and corresponding proportional sides of similar triangles.





**Figure 5.** Question 3.3 in the van Hiele test

As can be seen in S01's statement, properly stating proportional corresponding sides is a common difficulty among students.

S01 : *Because [of] Angle ACF and Angle ADE, their [ $\triangle ADE$  and  $\triangle ACF$ ] sides are proportional to each other. Assuming the side CF is proportional to segment DE and side AF is proportional to segment AE.*

S04 and S11 based their judgment on their observation and intuitive understanding of similar triangles:

S04 : *Parehas ng shape. Ang pinagkaiba lang yung laki. (The shapes are the same. The only difference is their size).*

S11 : *Because they have the same angles, they only differ in size.*

#### **Level 4 - Formal Deduction**

The first proving item was on triangle congruence. In [Figure 6](#), S14 was able to give correct statements except in statement #3. The pairs of congruent corresponding angles do not follow from any of the preceding ones. The student incorrectly assumed that the two triangles are isosceles based on how the figures were drawn. Instead of pairing corresponding angles of the two triangles, S14 paired two angles of one triangle. The reason for concluding "vertical angles are congruent" was incorrectly referred to as the "Vertical Triangle Theorem." Moreover, the student stated "Triangle Addition Postulate" instead of using SAS to justify the triangle congruence.

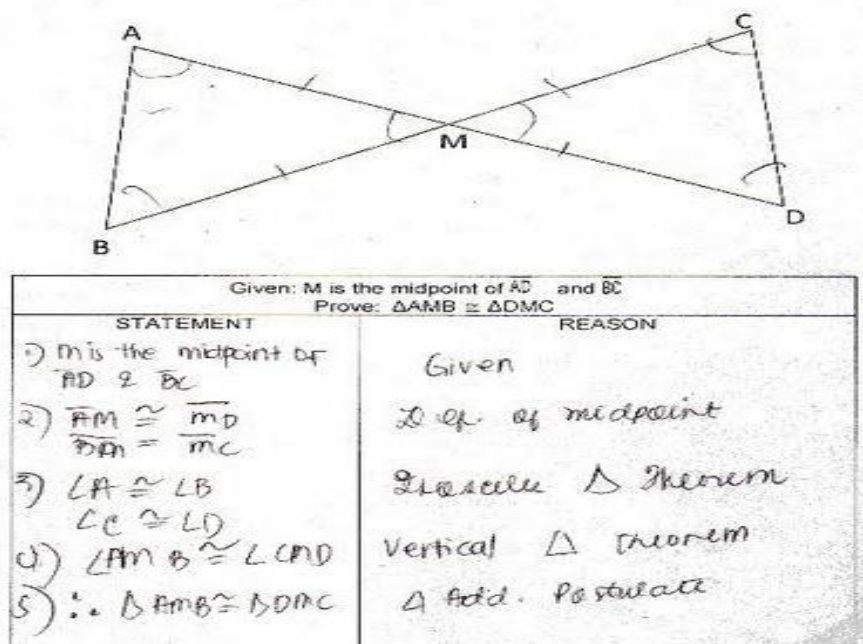


Figure 6. An Example of Proof with flaws by S14

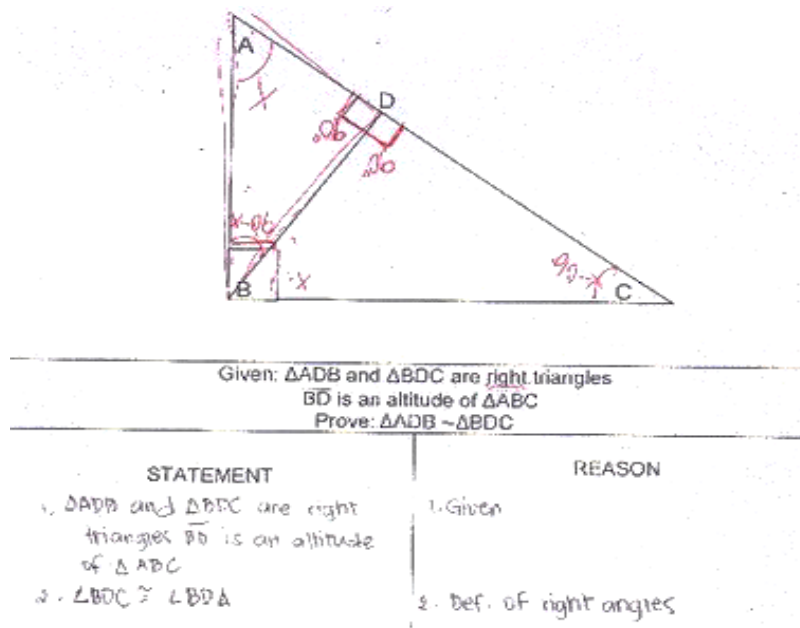


Figure 7. An incomplete proof by S06

The second proving item was on triangle similarity. Eight respondents gave incorrect proofs and two students did not answer this. Figure 7 shows an example of an incomplete proof. S06 correctly deduced the two right angles are congruent from the given right triangles. However, he could have first established that the angles BDC and BDA are right angles because the given triangles are right triangles. Instead, he incorrectly gave “definition of a right angle” (a right angle measures 90 degrees) as the reason. There was a missing statement before one can conclude that the angles are congruent – for example, angles BDC

and BDA are right angles. Moreover, the student did not know how to proceed (see Figure 7).

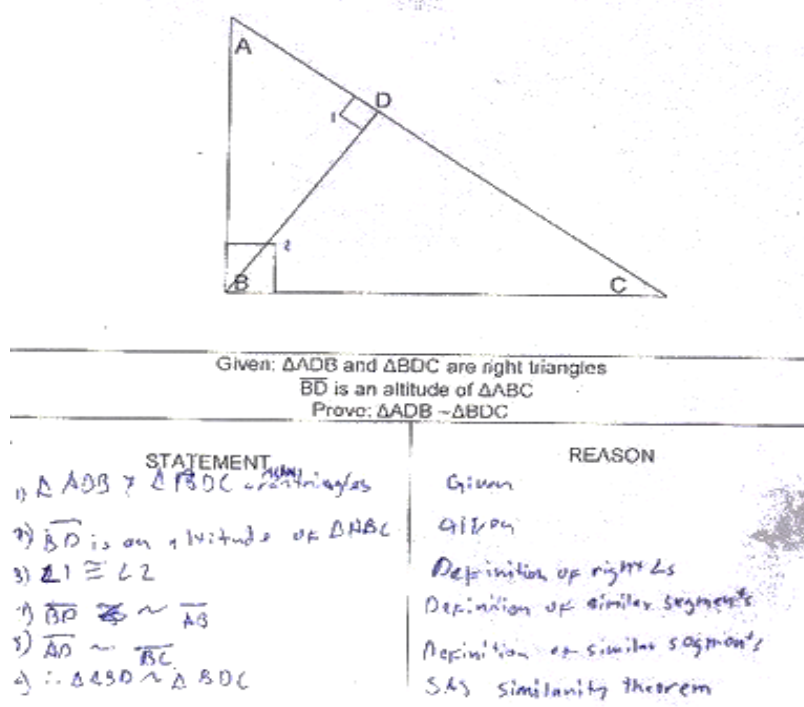


Figure 8. An example of proof with incorrectly stated reasons by S10

In the proof by S10 (see Figure 8), the right angle labeled 2 is not relevant in arriving at the desired conclusion as this is not an angle of either of the triangles under consideration. The student's use of pairs of similar segments is indicative of irrelevance as there is no such concept and this cannot be assumed from the given premises.

### 3.1.2. Students' Geometric Misconceptions

#### *Imprecise Terminology*

Students either had difficulty recalling the correct term or they simply mixed up the concepts. For example, a student defining an obtuse angle as having less than or equal to 90 degree measure, suggests that he was aware of the measure being in a range of values but got the different terms for the kinds of angles mixed up.

Students were not accurate enough to identify which reason was the most appropriate to justify their claim. For example, (1) the general Definition of Congruence was used when the student meant that the two triangles are congruent; (2) the Right Angle Theorem was used to refer the Pythagorean Theorem; (3) The Isosceles Triangle Theorem was used instead of the ASA triangle congruence; students chose a name that was close or quite related to what they were trying to prove.

Students had difficulty distinguishing between similar and congruent triangles. This may be due to students' difficulty in distinguishing between a conditional statement and its converse. Students' failure to identify the correct reason in their proofs may also be due in part to their failure to dissect or unpack the definitions, postulates, and theorems into premise and conclusion parts.

Students had a tendency to take chances and guess if they could possibly mention the correct justification to support their claim. Some students used “Postulate 7” and “Theorem 19” as reasons in their proofs. The tendency of students to use the numbered postulates and theorems had been found to be accepted by their geometry teachers during seatwork and exercises where the study was conducted. The effect of such a strategy to name a postulate or a theorem may be detrimental. Instead of focusing on reasoning, students tend to be preoccupied with memorizing the postulate or theorem number. It also raises doubts whether students actually understand the principle stated in the postulate or theorem. This seems to encourage guessing.

Some students consistently interchanged terms that were closely related. The term “equal” instead of “congruent” when describing the relationship between two triangles was a common error. Students were also confused when to say definition, postulate, and theorem.

Mathematical notations are symbolic representations of mathematical objects and processes. It is a language and a shorthand to communicate and express mathematical concepts and ideas. It has precise semantic meanings that are crucial in mathematical discourse. However, the study revealed students used improper notations. Similar segments were denoted as  $\overline{BD} \overline{AB}$  and  $\overline{AD} \overline{BC}$  even though there is no such concept of similar segments. The student must have meant sides are proportional which should be properly denoted as  $\frac{BD}{AD} = \frac{AD}{BC}$ .

### ***Identification/Classification of Basic Shapes***

Most respondents could easily identify triangles. Right triangles were easily identified when there is an indicated right angle (either with the perpendicular symbol or with the  $90^\circ$  measure). However, students failed to recognize shapes that involve additional concept and procedure such as those with the triangle angle sum, algebraic solutions and isosceles triangle theorem in shapes A and I of Question 1.2. Students gave naïve conceptions and informal understanding as they tended to judge the shapes by their mere appearance: with an L-shaped angle and the sides look like the hands of a clock at three o'clock. Students assumed that based on the drawing, the angle in the triangle looked like they were perpendicular even if neither the perpendicular symbol nor the  $90^\circ$  measure was written. This can be due in part to students' difficulty in determining what can or cannot be assumed, what was given and what was not.

### ***Parallelism and Perpendicularity***

Parallelism is one of the prerequisite concepts for understanding the principle of similarity. In question 3.3, a given a pair of parallel lines is cut by a transversal forming congruent corresponding angles. Another pair of congruent angles can be deduced by reflexivity. These two pairs of congruent angles make the triangles similar by AA triangle similarity. None of the respondents was able to recognize that each of the two sides of the larger triangle can serve as transversals of the parallel sides. Instead, some students showed informal understanding of similarity as indicated in their responses: “Triangle ADE is larger than triangle ACF...because the measure of the sides is larger” and “Triangle ACF expanded, become ADE”. Students' inability to apply the concept they previously learned from parallel lines to triangle similarity indicates that their knowledge seemed to be compartmentalized. To them, these concepts were not linked or connected to one another. It also connotes poor understanding as this knowledge from previous lessons were not retained and used when the situation called for it.

### ***Class Inclusion***

Misconceptions under this category are those that involve misunderstandings on the family of triangles. These are mostly caused by students' tendency to operate on properties exclusive to a specific type of triangle. Most of them can recognize a right triangle, but some of them cannot point out that a right triangle can also be isosceles. A minority of them said a right triangle cannot be isosceles and gave reasons that are not correct.

### ***Properties of a Triangle***

Recognition of the properties of a shape is necessary to relate shapes to each other. Misconceptions in this category are further classified into the following.

### ***Kinds of Triangles***

Students misunderstood the definition of isosceles triangle and its parts as seen in some responses such as "all angles of an isosceles triangle are equal" and "a base angle and its vertex angle are congruent." In another instance, students had difficulty handling the definition of acute and obtuse angles and the Triangle Angle Sum Theorem concurrently as seen in the following responses: "In an obtuse triangle, the sum of the two acute angles is equal to the obtuse angle." and "In an obtuse triangle, the sum of the two acute angles is greater than 90." This difficulty seems to show that students were seldom engaged or not engaged at all in higher order thinking discourse and problem solving.

### ***Angle Sum of Triangle***

Most of the students can only handle one or two but not all of the given conditions. For instance, in Question 2.3, a number of students overlooked the given conditions that an angle in the triangle is obtuse and the rule on sum of interior angles. They were fixated on the given two acute angles, disregarded the type of triangle being obtuse and said it is possible that the sum of these acute angles in the triangle is greater than 90°.

### ***Pythagorean Theorem***

The Pythagorean Theorem has been an important and popular concept in geometry. However, very few (5 out of 30) students were able to state this when they were asked to relate the three sides of a right triangle. Those who were able, answered the question correctly by naming the relationship in a procedural manner: "given a, b, and c as sides, a squared plus b squared is equal to c squared." It was not clear whether they referred to the side whose length is labeled c as the hypotenuse and the sides with lengths a and b, as the legs. This is worthy of note because this is fundamental to other succeeding concepts such as the Hypotenuse-Leg Triangle Congruence Theorem.

### ***Relationship between Two Triangles***

Students described congruent and similar triangles in their own words based on their mere appearance as opposed to analytically describing these in terms of their corresponding angles and sides. Students had informal understanding that similar triangles have the same shape but different sizes as most of them said "congruent triangles are not similar triangles." Students failed to recognize the difference between triangle congruence and similarity.

Plausible causes are their lack of knowledge of conditional statement and its converse, and failure to identify the necessary and sufficient conditions for each relationship.

Aside from the misconceptions that contributed to students' difficulty in proving, students lacked the necessary cognitive strategies to proceed from the premises. Some tacit premises in the figure that were relevant to use in proving were not recognized by some students: congruent angles due to reflexivity, vertical angles that are congruent, transversal line cutting parallel lines that form congruent angles, among others. Instead, irrelevant information which was not given and could not be assumed (congruence of angles in Statement #3 in Figure 6 and Figure 8, and proportionality of sides in Statements 4 and 5 in Figure 8) was used.

### 3.2. Discussion

Consistent with the findings in other international studies (Atebe, 2008; Luneta, 2015), less than 24% of the students reached the formal deduction stage. The percentage of Grade 9 students who reached the Level 4 is a slightly better than that of Grade 6 students (20.7%) in Taiwan (Ma et al., 2015) and that of another local study in the South of the Philippines by Solaiman, Magno and Aman (2017) with no single Grade 9 respondent who reached the informal deduction (Level 3). This implies that most were clearly behind their expected van Hiele levels after taking Euclidian Geometry in Grade 9. More concerning is that more than half of the respondents were still operating in Level 1. Aside from the general notion of students' poor geometric thinking, the study described detailed errors committed mostly in levels 2 and 3. Results in this study points toward how geometry is being taught in lessons that deal with analysis (level 2), abstraction (level 3) and eventually formal deduction (level 4).

The study explored, revealed, and described students' different misconceptions and difficulties in learning triangles. The use of imprecise terminologies could be caused by poor understanding of definitions, unmindful use of terms or preference of using informal language. While the use of informal language can help students gain intuitive understanding, eventually the regular use of the proper mathematical language and notation in textbooks and classroom discourse should be encouraged. The use of notation was introduced to lessen the use of texts and words. Yet, students take for granted the proper use of notation, this very thing that makes mathematics less cumbersome and teachers underrate students' difficulty of acquisition of notations in students' learning (Edwards, 2000). Language and notation play an important role in the development of conceptual understanding since instructional materials, resources, mathematics textbooks, and the like, use these in concept development. Knowledge of the correct technical terms and notations is necessary for learners to be able to communicate their ideas clearly and for them to be receptive of class discussions (Atebe & Schafer, 2010).

Most of the participants exhibited a lack of knowledge on geometric properties. This was due to either students lack communication skills in expressing their ideas or students knew the names of the various definitions, theorems, and postulates but did not know what they meant.

Students also had difficulty handling a lot of concepts and relating them to one another. Students' knowledge was mostly compartmentalized and they failed to see which concept is relevant to use to defend their answers. Most of the students in this group had superficial understanding and seem to regard geometry as a collection of unrelated concepts, rules, and properties. This may also explain why for them, mathematics in general is a difficult subject because concepts are interrelated. Since each concept is built on another, failure to master a previous concept adds to greater difficulty in understanding the

succeeding concepts. To say the least, a surface acquisition of a concept does not guarantee its recall and application in an unfamiliar problem.

Results of the study support Radatz's (1979) claim that learners' misconceptions are due to semantic differences between natural language and mathematical language, limited spatial abilities, failure to master the prerequisites, incorrect associations, lack of cognitive control and strategies, and application of rules or irrelevant ideas.

#### 4. CONCLUSION

On the basis of the results obtained, Grade 9 students have not attained the desired learning competencies expected of their level as far as the triangle concept is concerned. They are not ready to learn the concepts intended for the Grade 9 curriculum – quadrilaterals, its classifications and properties since they have not reached the van Hiele's formal deduction level on triangles. An in-depth analysis of their answers also reveals various misconceptions held by students on triangles. Students do not use the correct terminologies, mixed up concepts, write incorrect notations, grappled with simultaneous properties in a single figure, failed to connect previous concepts on parallel and perpendicular lines in triangles, and can't use tacit premises in their reasoning.

Consequently, we first recommend remedial work in geometry for 9th grade students, starting with Level 2 of the van Hiele model may be incorporated to the syllabus. Only once this is mastered should educators proceed to Level 3. It should not be assumed that just because students receive instruction in geometry at a young age that they come to 9th grade with a firm grasp of the ideas.

The manner by which the topics in geometry are being taught in the earlier grades needs to be re-examined. Some of these misconceptions may be deeply seated from early grade instruction. Also, Lim (2011) asserts that a major cause of misconception is in the communication line between the sender (teacher) and the receiver (student) when they operate at different van Hiele levels. For example, a teacher gives examples of two triangles having the same shape but different sizes and says that these are similar triangles. This may inadvertently contribute misconception in students' overgeneralization that for two triangles to be similar, they have to be of different sizes for as long as they have the same shape. Teachers' given examples and those not given can cause students' misconception (Bamberger et al., 2010). Teachers could form learning communities to share their list of students' misconceptions and discuss effective ways to counter or correct misconceptions.

Future studies on the van Hiele levels can include investigations in geometry topics other than the triangles and their properties, development of comprehensive test that can assess a wider scope of a particular geometric concept, and interventions to improve students' reasoning skills. Both pre-service and in-service teachers may also be assessed for their van Hiele levels and misconceptions. If teachers hold misconceptions, they are more likely to be unable to recognize errors students make and their instruction may inadvertently perpetuate these misconceptions (Graeber et al., 1989).

The van Hiele theory models the hierarchical property of geometric understanding. As noted by van Hiele (Atebe, 2008) the movement between two levels is not natural but undergoes a formal teaching-learning process and depends on the factors within the direct control of the teacher and the curriculum (Senk, 1989). Learners go through visualization before formal definitions, postulates, and theorems are introduced. As suggested by Barrett and Clements (2003) and Groth (2005), teachers can adapt the phases of learning: Information, Guided Orientation, Explication, Free Orientation, and Integration, in conscientious preparation of their instructional plan. Rather than presenting the lesson in a

transmissive manner, teachers may employ strategies to help learners be more prepared to do complicated and challenging tasks.

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