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MATHEMATICS TEACHER EDUCATORS' NOTICING OF PEDAGOGICAL CONTENT KNOWLEDGE ON HIERARCHICAL CLASSIFICATION OF QUADRILATERAL

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ABSTRACT

This study aims to investigate mathematics teacher educators' (MTE) knowledge in noticing preservice teachers' pedagogical content knowledge (PCK) on the hierarchical classification of the quadrilateral. A multiple case study was conducted to analyze the responses of ten MTEs in an online moderated-forum group discussion (M-FGD) from their written work on the MTE-PCK test completed prior to the M-FGD. The PCK test consisted of two tasks: the task that examines MTEs' knowledge to predict pre-service teachers' reason in representing the hierarchical classification of quadrilateral in Venn diagrams, and the task that examines MTEs' knowledge in making a flowchart as a recommendation to mathematics teacher to analyze the validity of quadrilateral classification. Results show that the MTEs indicate two considerations of noticing pre-service teachers' PCK on the quadrilateral classification: by definition and properties of quadrilaterals and by the visual appearance of quadrilaterals. Despite this, 20% of them were indicated to perform a lack of understanding of the hierarchical classification of quadrilaterals, as indicated by invalid flowcharts of validating the hierarchical classification of the quadrilateral.

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1. INTRODUCTION

Pedagogical Content Knowledge (PCK) has become one of the focuses of educational research in the last decade. Initially developed by Shulman (1986) with seven categories of knowledge as the basic knowledge for teachers to teach, several researchers have then developed various frameworks to understand the knowledge teachers need to teach (e.g., Chick et al., 2006; Rowland, 2013). Chick et al. (2006) developed a PCK framework that identifies various fields of knowledge in three different categories: clearly PCK, content knowledge used in a pedagogical context, and pedagogical knowledge used in a content context. Meanwhile, Rowland (2013) uses a framework called "the knowledge quartet" which consists of four aspects: foundation, transformation, connection, and contingency, to identify teachers' PCK. The framework has been used to interpret teacher PCK through quantitative and qualitative exploration and observing teaching practice (Maher et al., 2022; Mishra, 2020). It has been proven to be effective as a means of identifying the PCK of mathematics teachers.

Despite being a relatively new area of study, the knowledge needed by mathematics teacher educators to teach PCK to their prospective teachers has seen substantial advancement in recent years. Several researchers have also proposed various analytical frameworks, such as Chick and Beswick (2018) whose analytical framework describes an analogous relationship between PCK for Mathematics Teachers' PCK for MTE (MTEPCK). This framework was adapted from the PCK framework for Mathematics Teachers to the MTEPCK framework. Some researchers use this framework to explore MTEPECK, such as that done by Amador et al. (2021) to characterize the noticing of mathematics teacher educators (MTE) and their ability to interpret students' thinking, Pascual et al. (2021) to analyze MTE pedagogical knowledge specifically on the topic of symmetry, and Spangenberg (2021) who investigated the MTE-PCK of trigonometry. These three studies are found important as they serve as the basis for mathematics teacher educators' practice, including classroom-based examples and strategies, and give potential routes for mathematics teacher educators to undertake their own research on the nature and effectiveness of their course activities that promote and enhance prospective teachers' pedagogical topic knowledge.

While research on MTE-PCK has been emerging on the aspects of students' mathematical thinking and specific content in school mathematics, this type of knowledge is still scarce to be investigated in terms of geographical area. In the context of China, for example, the development of Chinese teachers' MTE-PCK has been found as one of the successes of the strong collaborative work of the mathematics education research team, which are also noted as mathematics teacher educators (MTE) (Paine et al., 2015), and mathematics teachers (Wu & Cai, 2021) to design and implement mathematics lesson although these findings need to be further studied in some untouched region in China (Wu & Cai, 2016). However, in Indonesia, the studies reported on mathematics teacher knowledge reported are still around the PCK of teachers, instead of the MTEPCK. Numerous research has been investigated to assess teacher knowledge and how it affects teachers' teaching practices, one of which indicates the Indonesian teachers' unsuccessfulness in integrating their PCK into the teaching practices due to the lack of PCK (e.g., Ekawati et al., 2015; Yunianto et al., 2021), or inconsistency between the PCK and their actual teaching (e.g., Muhtarom et al., 2019; Siswono et al., 2017). Thus, a deeper understanding of the underlying reasons for these findings needs to be investigated primarily on the teachers' engagement and experiences in teacher education, where mathematics teacher educators (MTEs) play a significant role in shaping the teachers' PCK (Oates et al., 2021).

In this regard, the framework of PCK is then further developed to identify the PCK of teacher educators, including those in Indonesia, which in this case, is known as mathematics teacher educators (MTE)' PCK. Zaslavsky and Leikin (2004) describe the MTEs' knowledge as knowledge possessed by mathematics teachers, namely knowledge of school mathematics. Meanwhile, the PCK of MTE is the PCK needed by MTE to introduce PCK to prospective mathematics teachers. This is in accordance with the term PCK for mathematics teacher educators or Mathematics Teacher Educator Pedagogical Content Knowledge (MTEPCK) by Chick and Beswick (2018). As teacher educators, MTEs' knowledge should be more than the knowledge teachers need to help students learn mathematics. This means, just as teachers need certain knowledge in teaching mathematics, MTE also requires certain knowledge to teach school mathematics to prospective teachers. This is in line with Zopf (2010) who states that teachers' knowledge is part of educators' knowledge. However, Beswick and Goos (2018) believe that the conceptualization of MTE knowledge is influenced by the school mathematics teacher's model of knowledge.

In relation to PCK, MTE knowledge can be investigated on a specific topic related to two-dimensional figures as one of the most important topics in school mathematics. By understanding the PCK of two-dimensional figures, it is arguable that the instrument measuring MTEs' knowledge of this PCK can be better constructed. For example, the PCK states that teachers need to recognize that a definition of a quadrilateral includes only necessary and sufficient conditions for this quadrilateral, identify several equivalents and non-equivalent definitions for the same quadrilateral, and understand the definitions of quadrilaterals that are more appropriate for younger students. Those pieces of knowledge can be transformed into MTE knowledge by thinking of what are the same and the differences between MTEs' noticing and prospective teachers' noticing skills. In this sense, Beswick and Goos (2018) noted that what teachers should notice (i.e., students' thinking) and what MTEs must eventually notice (i.e., teachers' thinking) focus on the thinking of specific groups with which they worked. While MTEs' noticing is to assist teachers in noticing students' mathematical thinking, they must be able to observe students' mathematical thinking in addition to teachers' thinking, specifically through interpretation and evidence. As interest in teacher noticing research has grown, so have conceptualizations of noticing. Jacobs and Spangler (2017) summarized the three versions of noticing component conceptualizations, beginning with attending only (e.g., Star & Strickland, 2008). They demonstrated the nested and integrated components of the two-component conceptualization, attending and interpreting (e.g., Goldsmith & Seago, 2011) and the three-component perspective, attending, interpreting, and deciding to respond (e.g., Goldsmith & Seago, 2011; Jacobs & Spangler, 2017).

In the case of quadrilateral, more particularly, rectangle remains a special case of a parallelogram in pre-service teachers' figural concepts leading them not to adopt the hierarchical relationship. The findings suggested that learners were likely to recognize quadrilaterals by a special case of them and prototypical figures, even though they knew the formal definition in general. This led learners to have difficulty understanding the inclusion relations of quadrilaterals in primary education, students are taught about the types of quadrilaterals by recognizing their shapes and studying their properties. In contrast, in advanced learning, a quadrilateral is a figure that has several special cases with regard to its properties. In order to teach quadrilateral, teachers should understand the relationship between quadrilateral properties and quadrilateral classification. In the context of quadrilateral classification, de Villiers (1994) states that two types of classifications can be made regarding the relationships between quadrilaterals: hierarchical classification and partial classification. With hierarchical classification, quadrilaterals are associated with one another within the framework of their properties as subsets (de Villiers, 1994). In other words, a quadrilateral can be said to be a special case as its properties are a subset of another quadrilateral's properties.

School mathematics classifies quadrilateral in a simple way using visual summarization. However, the above-mentioned classifications motivate further analyses in addition to simple and visual summarization of the information (Craine & Rubenstein, 1993) and require the establishment of appropriate relationships between concepts and images. It considers shapes as subsets of other shapes, so squares are seen as special cases of rectangles, and rhombi are included in the set of kites (Forsythe, 2015). In addition to the common approach of hierarchical classification, partial classification is used as an alternative to classifying the figures (de Villiers, 1994). In partial classification, quadrilaterals are independent of each other and, classified according to their properties as separate sets (Erez & Yerushalmy, 2006). Partial classification and definition are not acceptable in mathematical terms. They are simply partial, sometimes necessary and beneficial for a clear distinction between concepts (de Villiers, 1994). However, the partitional view can be held very strongly since it has been developed from "an early age", so students often find it difficult to accept the inclusion of some classes of shapes within others (Okazaki, 2009). The hierarchical classification involves comprehending the relationships between quadrilaterals, which is a rather difficult activity for many learners (Erez & Yerushalmy, 2006; Fujita, 2012). However, hierarchical classification allows MTE to deeply understand quadrilaterals and the relationships between their properties.

Hence, pedagogical content knowledge is said to be important for MTE to manage the learning activity such that it is suited to the pre-service teachers' needs. Furthermore, the MTEs' skill in attending, interpreting, and responding to prospective teachers' PCK (Simpson & Haltiwanger, 2017), or known as MTEs' noticing skill is arguably important for MTEs' professional knowledge. This skill is known as essential for teachers and MTEs since every aspect of teaching relies on notice (Mason, 2002), including seeing what students or PSTs are doing, how they are responding, comparing what is being responded to the standards of mathematics teaching and expectations, and predicting about what might be reacted next. Despite this, little focus has been given to those who teach teachers to notice, or in other words, research on how MTEs notice their prospective teachers' PCK is still scant. However, PCK can also depend on content knowledge possessed by an MTE. Additionally, understanding how MTEs notice their PSTs' PCK would give benefits in providing insights on expanding discussions either practically or theoretically on the enlargement of teacher-knowledge-related issues in teacher education. Hence, this study aims to explore MTEs' knowledge in noticing pre-service teacher knowledge on hierarchical quadrilateral classification.

2. METHOD

2.1. Research Design

This study used a multiple case study to observe mathematics teacher educators' (MTE) knowledge of noticing pre-service teachers' PCK of the hierarchical classification of quadrilaterals. A multiple case study is a type of case study that includes two or more cases to investigate the same phenomena (Lewis-Beck et al., 2003; Yin, 2017). The MTEs' knowledge of the hierarchical classification of quadrilateral was analysed using qualitative content analysis of the MTEs' response to the given question and repeated observations of the videotapes from online moderated forum group discussion (M-FGD) via an online meeting platform. The moderation process was led by the moderator, which is the first author. The moderation was held by optimizing the moderator's role in the discussion to maintain the rhythm of the discussion and reveal all aspects planned to be discussed in detail.

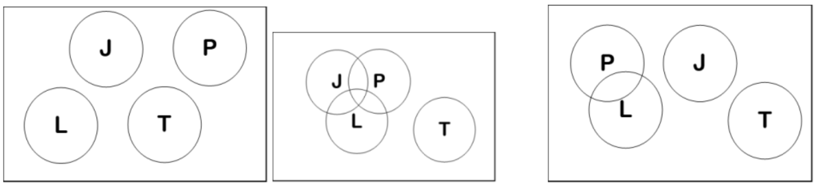
2.2. Sample

A total of ten MTEs with various backgrounds in terms of teaching experience in university, sexes, and background knowledge participated in this study. All the participants were Second-year students in Doctoral Mathematics Education program at a public university, in Surabaya, East Java, Indonesia. The participants were enrolled in a course, called Knowledge and Praxis of in-service and pre-service Teachers. The M-FGD was held at the beginning of the course where they did not learn about the content of the course in an explicit course related to teacher knowledge.

2.3. Research Instrument and Procedure

This study began with developing several item problems related to the hierarchical classification of a quadrilateral. A paper-based test was given to the research samples. Two tasks as shown in Figure 1 were given to explore MTEs' noticing of pedagogical content knowledge on the hierarchical classification of quadrilaterals. The first problem was asking MTE to predict (noticing practice) pre-service teachers' reason for drawing a Venn diagram about quadrilateral classification. Meanwhile, the second problem was asking MTE to draw a flowchart as recommendation guidance to validate pre-service teachers' Venn diagram.

1. Tiga mahasiswa calon guru matematika diminta untuk membuat diagram Venn yang menunjukkan himpunan segiempat.



Mahasiswa A Mahasiswa B Mahasiswa C

Keterangan: jajargenjang (J), layang-layang (L), persegipanjang (P), dan trapesium (T).

a) Tuliskan kemungkinan alasan mahasiswa A, B, dan C membuat diagram Venn tersebut.
b) Diberikan masing-masing sebuah jajargenjang (J), layang-layang (L), persegi panjang (P), trapesium (T), dan belah ketupat (B). Susunlah sebuah diagram alir *(flowchart)* yang dapat Anda rekomendasikan kepada mahasiswa calon guru untuk digunakan sebagai panduan dalam menganalisis kevalidan dari klasifikasi dalam bentuk diagram venn pada kelima bangun datar tersebut. Lengkapi diagram alir Anda dengan penjelasan yang memadai.

Translation:
Three prospective mathematics teachers are asked to draw Venn diagrams showing the classification of quadrilaterals.
Note: parallelogram (J), kite (L), rectangle (P), and trapezoid (T).

a) Write down the probable reasons that might underly each student: students A, B, and C create the diagram.
b) Given a parallelogram, a kite, a rectangle, a trapezoid, and a rhombus. Create a flowchart you may recommend to your students (the prospective teachers) that can be used as a guideline to assess the validity of the classification of those five quadrilaterals which is in the form of Venn diagrams. Complete the flowchart with any relevant explanation.

Figure 1. Hierarchical classification of quadrilateral tasks for MTE

To confirm MTEs' responses to the given problem and to explore more about MTEs' noticing pedagogical content knowledge on hierarchical classification of quadrilateral, M-FGD was held. The M-FGD was held for about an hour through a recorded online meeting.

A list of questions was created as a guideline for the moderators to organize the moderation process. The list of questions was made based on some MTE responses to the problem given before the M-FGD. The MTEs' response to the problem was analyzed by categorizing it into some categories based on similar responses. Then, the M-FGD was held to confirm the MTEs' response and explore the MTEs' knowledge of the hierarchical classification of quadrilaterals. The detailed list of questions is given in Table 1.

Table 1. Guidelines of the M-FGD

Problems	Questions
Predicting pre-service teachers' reason for making a Venn diagram about the hierarchical classification of quadrilateral	1. What is the reason for the pre-service teachers' answer? 2. Are there any quadrilateral properties that make pre-service teachers think that way? 3. Which parts of the quadrilateral's concept might not have been understood by students?
Make a flowchart as a recommendation to analyze the validity of quadrilateral classification	1. Which part can be used as a guide to analyzing the validity of the classification of quadrilaterals in the Venn diagram? 2. What is the meaning of the written flowchart? 3. How can the quadrilateral classification be validated using the written flowchart?

2.4. Data Analysis

Once the MTEs' responses were collected and the M-FGD was held, the samples' responses on the lists of questions in Table 1 were investigated based on Leinhardt and Greeno (1986) two fundamental systems of knowledge that must be mastered by teachers and teacher educators, which are subject matter and lesson structure knowledge. Subject matter knowledge is content knowledge to be taught to pre-service teachers (Muir et al., 2017). While lesson structure knowledge is knowledge to manage a lesson (Turnuklu & Yesildere, 2007). During learning activities, MTE must be able to predict and notice pre-service teachers' thinking processes, so that they can maintain learning activities according to the needs of participants. The research's aim is to observe mathematics teacher educators' (MTE) knowledge of the hierarchical classification of quadrilaterals by responding to three Venn diagrams about quadrilateral classification. The given Venn diagrams were about quadrilateral classification based on their special cases. Usiskin (2008) write the special types of quadrilaterals and their special cases as shown in Table 2.

Table 2. Special types of quadrilaterals and their special cases

Special types of quadrilaterals	Their special cases
Kite	Rhombus, Square
Trapezoid	Parallelogram, rectangle, rhombus, square
Parallelogram	Rectangle, rhombus, square
Rectangle	Square
Rhombus	Square
Square	Not available

In this study, data analysis was carried out concerning the MTEPCK framework by Chick and Beswick (2018) as shown in Table 3.

Table 3. MTEPCK category and indicator

Category	MTEPCK	Indicator
Clearly PCK: Students' thinking	Discusses or addresses PST's ways of thinking about an SMTPCK concept	MTE notice pre-service teachers' reasons for making Venn diagrams about the hierarchical classification of quadrilateral
Clearly PCK: Representation of concept	Describes or demonstrate ways to model or illustrate an SMTPCK concept	MTE create a flowchart as guidance to validate pre-service teachers' Venn diagram about the hierarchical classification of quadrilateral

While the framework of Chick and Beswick (2018) builds on existing research into PCK and categorizes aspects of the work of teacher education, this research only focuses on two aspects of the MTEPCK, namely PSTs' thinking and PSTs' representation of a concept.

3. RESULT AND DISCUSSION

In responding to the hierarchical quadrilateral classification problem, it is known that 80% of MTEs have a piece of knowledge and notice pre-service teachers' reason for drawing a Venn diagram related to the hierarchical classification of the quadrilateral. Meanwhile, 20% of MTE just re-describe the Venn diagram drawn by pre-service teachers. Then, MTEs' response was categorized into some categories based on the pre-service teachers' reason predicted by MTE. Table 4 shows the list of categories of MTEs' response.

Table 4. List of categories of MTEs' responses to the hierarchical classification of quadrilateral problem

List of categories	MTEs' response
Category 1	The pre-service teacher considers the definition and properties of a quadrilateral in a quadrilateral family
Category 2	The pre-service teacher considers the visual appearance of a quadrilateral
Category 3	Pre-service teacher lack understanding of the meaning and the properties of quadrilaterals

According to these results shown in Figure 2, MTE who said that the pre-service teacher considers the meaning and properties of quadrilaterals explained that parallelograms, rectangles, and kite are related to each other. As it is known that a rectangle is a parallelogram that has an angle of 90°, the pre-service teacher might understand that this only applies to some rectangles, so he drew a Venn diagram that intersects each other. In this regard, the MTE could notice that the pre-service teacher with this response can identify that trapezoid only has a pair of parallel sides. This shows that in predicting pre-service teachers' reasons for drawing such Venn diagram, MTE takes into account the pre-service teacher's knowledge of the special case of the quadrilateral which can be defined exclusively and inclusively (Usiskin, 2008). An exclusive definition is said to be true for a specific quadrilateral, while an inclusive definition is valid for a family of quadrilaterals. For example, the definition of a trapezoid which states that a trapezoid has only one pair of parallel sides is called the exclusive definition. While the inclusive definition of a trapezoid states that a trapezoid is a quadrilateral with at least one pair of parallel sides.

Mahasiswa B dapat melihat keterkaitan antara jajar genjang, persegi panjang, dan layang-layang. Ia dapat melihat bahwa suatu jajar genjang yang besar tiap sudutnya 90° adalah persegi panjang. Namun, ia melihat keterkaitan antara himpunan jajar genjang dan persegi panjang sebagai dua himpunan yang saling beririsan, yang artinya ia mungkin tidak melihat persegi panjang sebagai bentuk khusus dari jajar genjang atau bahwa semua persegi panjang adalah jajar genjang yang lebih tepat disajikan sebagai persegi panjang adalah subset dari jajar genjang pada diagram Venn tersebut.

Translation:
Student B could observe the relationship between rectangles, parallelograms, and kites. He could see that a parallelogram with a measure of 90 degrees is a rectangle. However, he saw the relationship between the sets of parallelograms and rectangles as intersecting sets, which means he probably did not see rectangles as a special form of parallelograms or that all rectangles are parallelograms which is more properly presented as rectangles are a subset of parallelograms on the Venn diagram.

Figure 2. Category 1 of MTEs' responses

Another reason mentioned by MTE is that the pre-service teacher makes a Venn diagram because the pre-service teacher only observes each shape of quadrilaterals that is different, such as the sides of the quadrilateral, the angles, or the diagonals. This means that the preservice teacher's knowledge of quadrilaterals is limited to exclusive definitions. In this case, category 2 is related to category 3 which states that the pre-service teacher experienced a misunderstanding of the definition and properties of a quadrilateral. However, several MTEs in the FGD stated that they did not agree that the pre-service teacher did not understand the quadrilateral concept. Pre-service teacher errors in making Venn diagrams can occur because of the different perspectives of pre-service teachers in classifying quadrilaterals, where the pre-service teacher has not been able to accept or understand the quadrilateral classification hierarchically, but partially. As Erez and Yerushalmy (2006) said, the partial classification of quadrilaterals is carried out based on the properties of each quadrilateral in a separate set, not interrelated. Hence, it can be said that the preservice teacher's knowledge of quadrilaterals is still limited to exclusive definitions (Josefsson, 2013). The limitations of the preservice teacher's knowledge can occur because, during the learning period, the pre-service teacher started studying the special forms of quadrilaterals and the characteristics of each shape. It is different from the inclusive definition which states that quadrilateral properties can also apply to quadrilateral families. To assess the pre-service teacher's understanding of the quadrilateral classification hierarchically, MTE was asked to make a flowchart as a guide to validating the pre-service teacher's answers. The results of this study indicate that 80% of MTE know to create a flowchart as recommendation guidance for validating the hierarchical classification of the quadrilateral. While the rest 20% of MTE do not know because they can't create the flowchart. There are two categories of MTE response to the second problem as shown in Table 5.

Table 5. MTEs' response to the 2nd problem

	Category 1	Category 2
MTEs' response	A quadrilateral flowchart that shows the special types of quadrilaterals and their special cases based on their properties and their explanation	A quadrilateral flowchart that shows the special types of quadrilaterals and their special cases based on their properties without explanation

Referring to Table 5 and the results of the M-FGD, some MTEs have difficulty in making flowcharts. This shows that MTEs' pedagogy content knowledge in making guidance to assess pre-service teacher answers to the hierarchical quadrilateral classification problem has not been achieved. Meanwhile, the other two categories provide a different flowchart description. Category 1 describes flowcharts that are written based on the properties of the quadrilateral family and its special case that leads to an inclusive definition of a quadrilateral. The flowchart made by MTE is shown in Figure 3 and started with quadrilateral in general, where if one of the angles in the quadrilateral is greater than 180°, it is called a concave quadrilateral. Meanwhile, if the interior angles in the quadrilateral are each less than 180°, it is called a convex quadrilateral. A convex quadrilateral that has at least one pair of parallel sides is called a trapezoid. If two pairs of parallel sides are the same length, it is called a parallelogram. Furthermore, a parallelogram in which each angle is 90° is called a rectangle. While a kite is a quadrilateral that has two pairs of adjacent sides that are the same length. However, if all sides are the same length, then the kite is called a rhombus. MTE also writes that a rhombus with an angle of 90° is a rectangle which according to its exclusive definition, is not quite right because the special case of a rhombus is a square (Usiskin, 2008). However, inclusively, this could also be true because a square is also a special case of a rectangle.

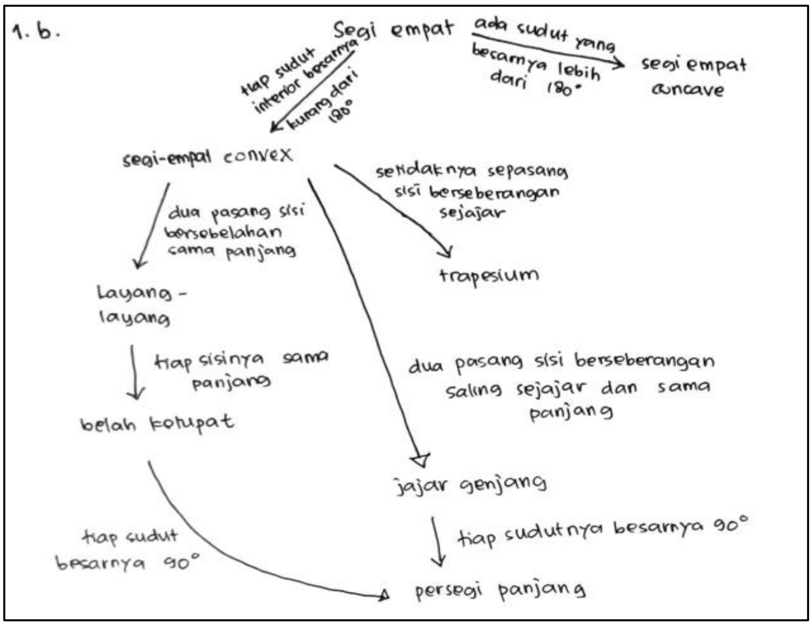


Figure 3. Category 2 of MTEs' response to the 2nd problem

On the contrary, category 2 shown in Figure 4 is a flowchart created based on the visual appearance of a quadrilateral, which is still related to the concept of quadrilateral, but without a detailed explanation. In this category, MTE explains that parallelograms, kites, and rectangles are different types of quadrilaterals because of their different visual appearances, such as the length of the sides, pairs of parallel sides, and the angles they have. Meanwhile, rhombi and trapezoids are special cases of rectangles. This shows that MTE has not been able to determine a hierarchical classification of quadrilaterals and MTEs' knowledge of quadrilaterals' definition is limited to an exclusive definition, so the guidance flowchart refers to the partial classification of quadrilaterals (de Villiers, 1994).

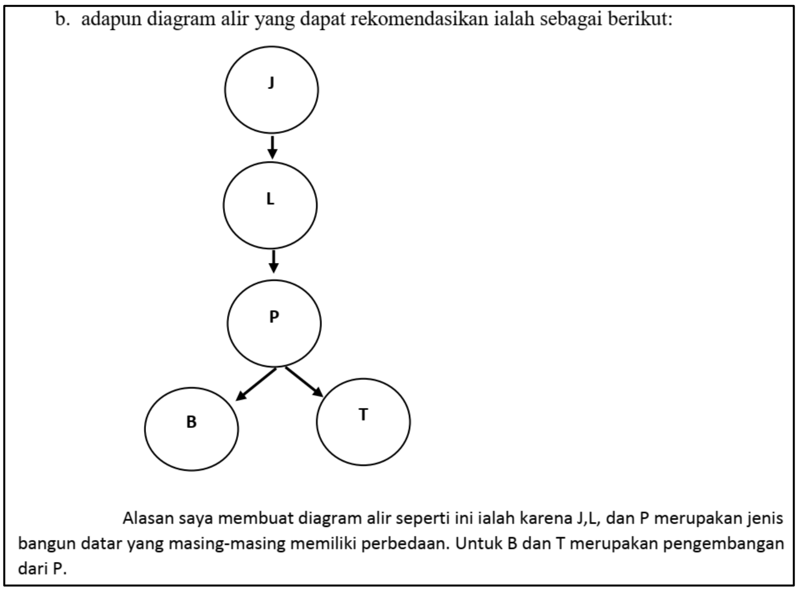


Figure 4. Category 3 of MTEs' responses to the 2nd problem

Based on the results of the M-FGD, several conclusions can be drawn as follows. MTEs' pedagogical content knowledge on the hierarchical classification of a quadrilateral is divided into some categories. Those who understood the hierarchical classification could explain pre-service teachers' reason related to the relation of quadrilateral properties, that some of them are subsets because of their special cases. However, in their understanding of the classification, although hierarchical, some were deficient in their concept images of parallelograms, as evidenced by the response in Figure 3, where this finding also occurred in preservice teachers (Birgin & Özkan, 2022). While those who have difficulties in understanding the hierarchical classification, use partial classification and refer to the quadrilaterals' different visual appearance. This difficulty can be explained by their incomplete informal deduction where there is a gap between their personal concept images and the personal concept definition of geometric figures (Fujita & Jones, 2007), which may lead them only depend on their visual perception of quadrilaterals, instead of properties of each fo quadrilateral (Leton et al., 2020). Pedagogically, to help students or PST achieve informal deduction, the MTE should be encouraged to understand that investigating lists of properties of quadrilaterals, making comparisons, and reaching abstractions such as 'if the property... holds for quadrilaterals... then it holds true for quadrilaterals...', as suggested by Fujita (2012), is necessary.

MTEs' noticing of pre-service teachers' hierarchical classification has an impact on MTEs' ability to draw a flowchart as a guideline for validating pre-service teachers' quadrilateral classification. The findings that some MTEs applied partial classification, instead of hierarchical classification, on quadrilateral when validating pre-service teachers' quadrilateral classification show that the MTEs have problems with their content knowledge which led them to fail in noticing PST's quadrilateral classification. While they included some evaluative comments in their professional noticing practices, they lack in-depth interpretative analysis of student thinking, as shown by their invalid classification, where this finding is also reported by Amador et al. (2021) who found that their participating preservice teachers rarely create links between student thinking and the broader concepts of teaching and learning.

As many previous researchers suggested (e.g., Avcu, 2022; Erdogan & Dur, 2014), the hierarchical classification of geometric figures requires the mastery of many aspects, some of which are the ability to identify the transitivity relation between concepts, asymmetry of the relations between quadrilaterals, and the asymmetry of the properties between quadrilaterals. According to Erdogan and Dur (2014), those three aspects can be seen when individuals label two quadrilaterals. In this study, the MTEs tried noticing PSTs' responses to the classification of quadrilateral by assessing the asymmetry of the relations or the properties between quadrilaterals, instead of the extent to which the transitivity relation emerges from PSTs' responses. For example, the responses indicated in Figure 2 and Figure 3 reveal that all squares are rectangles, but not all rectangles are squares, by identifying the properties of rectangle and square through the Venn diagram analysis. No indication was found in all the MTEs' responses that explicitly express the idea of using definition as the tool for transitivity analysis, like a response if a square is a rectangle and if a rectangle is a parallelogram, then a square is a parallelogram. Thus, the MTEs could identify the PSTs' difficulties related to the use of prototypical images in choosing a family category among the given quadrilaterals (e.g., Erdogan & Dur, 2014), even when individuals might know a precise definition (Fujita, 2012).

Responding to this study's findings, we would highlight that the hierarchical and partial classification of quadrilateral become essential knowledge for MTE to assess their PSTs' PCK on quadrilateral classification. According to de Villiers (1994), there is a rigid relationship between object definition and classification. For example, the trapezoid definition "a quadrilateral having at least one pair of opposite sides parallel" leads to a hierarchical classification in which the parallelogram is a particular case of a trapezoid; however, if the definition states that there is only one pair of opposite sides parallel, we have a partition classification, with the parallelogram belonging to a disjoint set of the trapezium. Thus, MTE should also notice the existence of whether the PSTs are working with an exclusive or inclusive definition of a trapezoid.

4. CONCLUSION

This research finds that MTEs have different views on classifying quadrilateral. Even though they have passed advanced education and are known as teacher educators, not all MTEs notice pre-service teachers' hierarchical classifications and define quadrilaterals inclusively. Hence, 20% of MTEs do not have any knowledge about pre-service teachers' ways of thinking on the hierarchical classification of quadrilaterals and failed to draw a guidance flowchart for validating pre-service teachers' hierarchical classification of a quadrilateral. Therefore, MTEs' knowledge of quadrilaterals, specifically the hierarchical classification of quadrilaterals, should be improved.

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