

Exploring the role of cognitive style and APOS-based scaffolding in developing students' advanced mathematical thinking on function limits

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Abstract

This qualitative case study explored how field dependence-independence shaped students' access to Action-Process-Object-Schema (APOS)-based scaffolding and how stage-specific support contributed to the qualitative development of advanced mathematical thinking (AMT) on function limits. In this study, developing denotes the observable emergence or reorganization of mathematical constructions during scaffolded activity, not a statistically measured learning gain. The bounded classroom case involved 24 mathematics education students at an Islamic university in Sumatra, Indonesia. The Group Embedded Figures Test classified 6 students as field-independent (FI) and 18 as field-dependent (FD); four information-rich cases, comprising 2 FI and 2 FD students, were selected for cross-case analysis. APOS-based scaffolding was implemented in four 135-minute Calculus I sessions and examined further through stimulated-recall interviews. Initial written solutions provided the primary evidence, post-prompt explanations and revisions documented students' responses to Action-, Process-, Object-, and Schema-oriented support, and classroom observation records verified the enactment and gradual fading of the planned scaffolds. FI-1 constructed a complete Schema and required only fading prompts. FI-2 moved from correct but disconnected Objects toward an integrated Schema after a Schema-oriented prompt. FD-1 corrected an invalid limit value after Process-oriented prompts stabilized the transformation sequence. FD-2 moved from direct substitution and $0/0$ toward guided algebraic Actions and an emerging Process after explicit Action-oriented cues. The findings indicate that cognitive style influenced task entry and the amount of external structure used, whereas APOS identified the construction that support needed to target. The observed shifts support a diagnostic account of AMT development while avoiding causal or long-term claims.

Keywords:

Advanced mathematical thinking, APOS theory, Field dependence-independence, Function limits, Scaffolding

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1. INTRODUCTION

Function limits constitute a conceptual gateway to continuity, differentiation, integration, and mathematical analysis. However, students often approach limit tasks as routines for substitution and algebraic manipulation rather than as problems that require reasoning about covariation, approximation, and values that are approached but not necessarily attained. Research involving university students and prospective mathematics teachers continues to identify fragmented concept images, confusion between a function value and a limiting value, weak coordination across representations, and difficulty interpreting the indeterminate form $0/0$ as a signal for further analysis rather than as a numerical result (Bansilal & Mkhwanazi, 2022; Borg & Fahlgren, 2026; Mirin, 2024; Tall & Vinner, 1981; Thompson & Harel, 2021). These difficulties have particular importance in mathematics teacher education. Prospective teachers must not only obtain a correct answer, but also identify the structure of an expression, justify each transformation, connect intermediate results, and communicate a mathematically defensible conclusion.

The central difficulty in learning limits is therefore not reducible to procedural accuracy. Understanding a limit requires students to coordinate processes, reorganize familiar algebraic relations, and treat the resulting limit as an object that can be examined and used in a broader argument (Cottrill et al., 1996). Advanced mathematical thinking (AMT) provides an appropriate framework for examining this transition from isolated operations to connected mathematical reasoning. Tall (1991) characterizes AMT through abstraction, conceptual reorganization, and formal reasoning. In the present study, abstraction refers to recognizing and reusing an underlying relation, such as a difference of squares or a conjugate structure. Conceptual reorganization refers to coordinating separate transformations and intermediate values into a coherent solution structure. Formal reasoning refers to constructing a logically connected justification for the transformations, limit evaluations, and final identity. These dimensions are treated as qualitative indicators of mathematical activity, not as independent psychometric scores.

Students may display these forms of reasoning through different patterns of information processing. The field dependence-independence framework concerns the extent to which learners can disembed relevant information from a surrounding context and reorganize it through an internally generated reference structure (Witkin et al., 1977). Field-independent (FI) students tend to identify and restructure relevant information more autonomously, whereas field-dependent (FD) students may rely more strongly on explicit organization, instructional cues, or interpersonal guidance. This distinction does not represent a hierarchy of mathematical ability and should not be interpreted as a basis for assigning a fixed teaching method to a cognitive-style category. Recent criticism of mechanical instructional matching further supports the use of cognitive style as an analytical lens for examining task entry and access to support, rather than as a deterministic learner label (Hattie & O'Leary, 2025).

APOS theory provides a complementary explanation of the mathematical constructions that students form while working with limits. In APOS theory, an Action is an externally prompted transformation; a Process emerges when the learner can mentally coordinate and reverse related actions; an Object is formed when a process can be examined,

validated, and reused; and a Schema integrates relevant Actions, Processes, Objects, and their relations into a coherent conceptual structure (Arnon et al., 2014; Dubinsky, 2014). For function limits, these constructions may include direct substitution, factorization, conjugate rationalization, cancellation, evaluation of the resulting expression, and integration of separate limit values into a final proof. APOS-based scaffolding can support these constructions through targeted cues, comparison questions, sequencing prompts, requests for explanation, and questions that require students to validate and reuse completed results. Research indicates that APOS-informed activities can strengthen conceptual understanding in university mathematics, while the effectiveness of scaffolding depends on the learner's available resources and on the specific mechanism targeted by the support (Borji et al., 2022; Chien et al., 2026; DiNapoli & Miller, 2022; Hofer & Reinhold, 2025; Martínez-Planell et al., 2024).

Despite these contributions, the relevant literature remains theoretically and empirically fragmented. Studies of function limits primarily document conceptual and representational difficulties. APOS studies identify the mental constructions required for advanced mathematical concepts. Scaffolding studies explain how contingent support can sustain strategy selection and reasoning. Cognitive-style research describes differences in how learners organize information and use external structure. Limited evidence connects these perspectives within the same instructional episode. In particular, little is known about whether FI and FD students differ mainly in selecting an initial Action, coordinating a Process, validating an Object, or integrating a Schema; how stage-specific APOS prompts change their subsequent reasoning; and how such prompt-response changes appear in abstraction, conceptual reorganization, and formal justification. Analyses based only on final correctness cannot reveal these mechanisms.

This study addresses that gap through an integrated diagnostic model. Cognitive style is used to examine how students initially identify, organize, and act on information in the same function-limit task. APOS theory is used to locate the construction that is already available, remains unstable, or emerges after support. AMT is used to describe the quality of the resulting mathematical reasoning. Within this model, APOS-based scaffolding does not replace students' reasoning or prescribe identical support for all members of a cognitive-style group. Instead, it targets the next mathematical construction indicated by the student's written work and interview response. The term *developing* in the title therefore refers to observable, short-term qualitative change in a student's mathematical construction during scaffolded activity, such as recognizing a previously unnoticed relation, coordinating a valid sequence of transformations, validating a limit value, or integrating separate results into a coherent argument. It does not claim a statistically measured gain, a causal treatment effect, or long-term development.

Accordingly, this qualitative cross-case study asks: (1) How do FI and FD students initially organize and solve an integrated function-limit task? (2) How do students with different cognitive styles respond to Action-, Process-, Object-, and Schema-oriented scaffolding, and what changes in APOS construction become visible? (3) What patterns of development in abstraction, conceptual reorganization, and formal reasoning emerge across the four cases?

The study offers three contributions. Theoretically, it connects cognitive style, APOS, scaffolding, and AMT within one explanatory framework. Methodologically, it links students' initial written work with prompt-response interview evidence rather than inferring development from final answers alone. Pedagogically, it proposes stage-specific diagnosis as a basis for scaffolding, avoiding uniform prescriptions for FI or FD students.

2. METHOD

2.1. Research Design and Context

The study used a qualitative, bounded case study design to examine connections among cognitive style, APOS-oriented support, and students' mathematical reasoning within one instructional context (Baxter & Jack, 2008; Yin, 2018). The case was bounded by the teaching of function limits in a Mathematics Education Program at an Islamic university in Sumatra, Indonesia, during the even semester of 2024/2025. The unit of in-depth analysis was each selected student's developmental pathway from the initial written response to the explanation or revision produced after stage-specific scaffolding. The design was explanatory and comparative. It examined qualitative change within cases and contrast across cases without testing causal effects or statistical hypotheses.

2.2. Participants and Cognitive Style Classification

The bounded classroom case involved 24 mathematics education students who had previously studied introductory differential calculus. Before the APOS sequence, cognitive style was classified using an Indonesian-language paper-and-pencil version of the Group Embedded Figures Test (GEFT), originally developed by Witkin et al. (1971) to examine field dependence-independence through the ability to locate a simple figure embedded in a complex visual configuration. The adapted booklet retained the original visual figures, item order, response procedure, and scoring structure, while the written instructions were presented in Indonesian. The GEFT contains three sections. Section I includes seven practice items and is not scored. Sections II and III contain nine scored items each. Each correctly traced figure receives one point, whereas an incorrect or unanswered item receives zero, producing a total score from 0 to 18. The complete classroom administration, including instructions and practice, lasted 20 minutes, consistent with the time allocation stated in the Indonesian booklet. Higher scores indicate a stronger field-independent tendency. For contrasting-case classification, scores of 0–11 were categorized as field-dependent and scores of 12–18 as field-independent, following an operational classification used in prior mathematics education research (Mirlanda & Pujiastuti, 2018). The cutoff was used for purposive grouping and case selection only; it was not treated as a measure of mathematical ability or as an absolute psychological diagnosis.

GEFT classified 6 students as FI and 18 as FD. Four information-rich cases were purposively selected for contrasting-case analysis: FI-1, FI-2, FD-1, and FD-2. Selection considered cognitive-style category, completeness of written work, and the potential to illuminate different solution pathways. The balanced four-case subsample supported within-

group and cross-group comparison. It was not intended to represent the 24 students statistically or to claim thematic saturation (see Table 1).

Table 1. Participant and data-source structure

Research layer	Number	Role in the study	Data generated
Bounded classroom case	24 students	Participants in four APOS-oriented sessions	Observation records, initial written tasks, and learning documents
In-depth case layer	4 students	Contrasting cases: FI-1, FI-2, FD-1, and FD-2	Initial solutions, scaffolded interview responses, and case profiles
Cognitive-style classification	24 students	GEFT grouping: 6 FI and 18 FD	FI and FD classifications used for case selection

2.3. APOS-Based Scaffolding and the AMT Elicitation Task

APOS-based scaffolding was implemented across four Calculus I sessions on function limits. Each session lasted 135 minutes. The course lecturer delivered the sequence using a study-prepared guide, while the researcher observed enactment and collected student work. Action-oriented prompts focused attention on the source of $0/0$ and on transformations such as factorization or multiplication by a conjugate. Process-oriented prompts asked students to explain, order, and predict transformation sequences. Object-oriented prompts required students to validate completed limit values and reuse them as mathematical entities. Schema-oriented prompts required integration of both limit results into a coherent justification. The same stage-specific logic guided the stimulated-recall interviews. Prompts began at the APOS construction indicated by the initial written response and were reduced when the student could continue independently. Scaffolds supplied cues and questions, not completed solutions (see Table 2).

Table 2. APOS-oriented scaffolding used in function-limit learning

APOS construction	Instructional purpose	Example scaffold	Evidence sought
Action	Select and execute a valid transformation.	What produces $0/0$? Which factorization or conjugate exposes a common factor?	Correct factorization, rationalization, and simplification.
Process	Interiorize and coordinate a transformation sequence.	Why is cancellation valid? Predict the simplified form before substitution.	Connected explanation and stable coordination of recurring transformations.
Object	Validate a completed limit value and reuse it as an entity.	How can the values of a and b be checked and then reused?	Validated values treated as completed mathematical objects.
Schema	Integrate both limits into one coherent argument.	How do a and b establish $a^2 + 3b^2 = 8$?	A connected justification linking both limits and the final identity.

An integrated, researcher-designed task served as the common elicitation context. Students determined $a = \lim_{x \rightarrow -1} \frac{x^2 - 1}{\sqrt{3x+4} - \sqrt{x+2}}$ and $b = \lim_{x \rightarrow 2} \frac{\sqrt{x+1} - \sqrt{9-3x}}{x-2}$, and then verified $a^2 + 3b^2 = 8$. The task required two limit evaluations and one integrated justification. Students first produced an unaided written response. The lecturer then used the response during a stimulated-recall interview to identify the point of difficulty and provide the least intensive APOS-aligned prompt needed for continuation. The task elicited qualitative reasoning and short-term response to scaffolding; it did not generate a numerical AMT score.

2.4. Data Collection

Three data sources were used with an explicit hierarchy. The four students' initial written solutions were the primary evidence because they displayed task entry, algebraic transformations, intermediate results, and final integration before interview support. Semi-structured stimulated-recall interviews then examined how students interpreted their own work and responded to stage-specific APOS prompts. For each case, the researchers documented the initial difficulty, the prompt delivered, the student's subsequent explanation or revision, and the degree of independence shown. Representative excerpts were selected when they directly clarified strategy choice, the point of breakdown, the effect of a prompt, or the student's ability to continue after support was reduced. The excerpts reported below are English translations of the Indonesian interview responses. Classroom observation was recorded on a structured sheet that logged the session, APOS focus, lecturer prompt, visible student response, degree of independence, and evidence of scaffold fading. The four observation records were coded OBS-S1 through OBS-S4. Reflection notes and learning materials provided supplementary evidence. Claims about initial mathematical construction were grounded in written work, claims about immediate development were grounded in the comparison between the initial response and the documented post-prompt response, and claims about implementation were checked against the observation record for the corresponding APOS session.

2.5. AMT Elicitation and Qualitative Coding

AMT was analyzed qualitatively through three dimensions derived from Tall's framework: abstraction, conceptual reorganization, and formal reasoning. The open-ended task was an elicitation instrument, not a standardized AMT test. For each dimension, initial evidence was coded as not evident, emerging, or explicit using the observable criteria in [Table 3](#). A developmental response was then recorded when stage-specific support enabled the student to identify a previously unrecognized relation, coordinate a previously unstable sequence, validate a result, or integrate completed values. Such changes were described as assisted emergence or consolidation, depending on the degree of independence. The codes were qualitative descriptors and were not converted into scores, summed, averaged, or compared statistically.

Table 3. Qualitative coding framework for AMT in function-limit learning

AMT dimension	Observable indicator in limit learning	Primary evidence	Coding interpretation
Abstraction	Identifies and reuses an underlying relation, such as factorization or a conjugate structure.	Written solution and interview context	Not evident: isolated operation. Emerging: relation partly recognized or used. Explicit: relation identified, justified, and reused coherently.
Conceptual reorganization	Coordinates symbolic actions and intermediate values into a coherent structure.	Written solution, supported by interview and observation context	Not evident: steps remain disconnected. Emerging: partial or unstable coordination. Explicit: coherent integration of relations and results.
Formal reasoning	Constructs a logically connected justification for transformations, limit values, and the final identity.	Written solution and interview context	Not evident: conclusion without justification. Emerging: reasoning is partly connected. Explicit: a coherent sequence of justified claims.

2.6. Data Analysis and Trustworthiness

Data were examined through a framework-guided cross-case analysis. The researchers first coded each initial written solution for task entry, interpretation of 0/0, factorization, conjugate rationalization, cancellation, formation of limit values, and final integration. Interview records were then coded as a prompt-response sequence: initial APOS position, scaffold type, post-prompt action or explanation, revised APOS position, and change in AMT evidence. Observation records were coded by session and examined for five implementation indicators: correspondence between the planned APOS focus and the lecturer prompt, visibility of the intended mathematical action, student response to the prompt, reduction or continuation of support, and consistency with the collected written work. GEFT category, APOS construction, and the three AMT dimensions served as sensitizing concepts rather than variables for hypothesis testing. Case profiles were constructed for FI-1, FI-2, FD-1, and FD-2 and compared within and across cognitive-style categories through iterative coding, review, and theme refinement (Braun & Clarke, 2006).

Trustworthiness was supported through source and method triangulation, a prompt-response evidence matrix, and retention of disconfirming evidence (Nowell et al., 2017). The matrix linked each claim to the initial written step, the scaffold delivered, the observed response, the APOS interpretation, and the AMT code. Within-category variation was retained. FI-2, for example, remained a contrasting FI case because autonomous strategy selection did not initially produce an integrated Schema. Developmental claims were limited to changes explicitly documented during the scaffolded interaction. The analysis did not treat supported performance as evidence of independent mastery, retention, or transfer.

2.7. Ethics

Ethical approval was obtained from the Institute for Research and Community Service (*Lembaga Penelitian dan Pengabdian kepada Masyarakat*, LP2M) under approval number STM/031/UI-AN/LP2M/III/2025. Participation was voluntary. Participants received information about the study, provided written informed consent, and were told that

participation or withdrawal would not affect course grades. Codes FI-1, FI-2, FD-1, and FD-2 replaced personal identifiers in transcripts, analytic files, figures, and reporting. Digital and printed data were accessible only to the research team.

3. RESULTS AND DISCUSSION

3.1. Results

The cross-case analysis identified variation in initial task organization, APOS breakpoints, and responses to stage-specific scaffolding. The FI cases selected relevant transformations with less external structure, but differed in whether correct limit values were integrated. The FD cases required clearer cues, although the needed support began at different APOS constructions. The results distinguish the initial written pathway from the subsequent scaffolded response. Development refers to the observable reorganization produced during that interaction, not to long-term or statistically measured improvement. Short interview excerpts are presented with the written evidence to show how each student understood the initial difficulty and what changed after the APOS-oriented prompt.

3.1.1. Field-Independent Cases: Autonomous Restructuring with Different Levels of Explicit Reasoning

FI-1 produced the most complete initial pathway (see [Figure 1](#)). For the first limit, the student factored $x^2 - 1$ as $(x-1)(x+1)$, multiplied by the conjugate of the radical denominator, canceled the common factor, and obtained $a = -2$. For the second limit, FI-1 transformed $4x-8$ into $4(x-2)$, canceled $x-2$, and obtained $b = 2/\sqrt{3}$. The student then verified $a^2 + 3b^2 = 8$. During the interview, the lecturer used only justification prompts, such as asking why cancellation remained valid when x approached the excluded value. FI-1 explained that the variable approached but was not equal to the limiting value during simplification and continued without procedural assistance.

FI-1 explained, *"I first tried direct substitution. It gave zero over zero, so I knew the expression had to be transformed."* When asked to justify cancellation, the student added, *"The variable approaches -1 but is not equal to -1 during simplification, so the common factor can be cancelled."* These statements confirmed recognition of the indeterminate form and conceptual control of the transformation rather than mere procedural imitation (Lithner, 2008, 2017).

Jika diketahui ← Given

a. $\lim_{x \rightarrow -1} \frac{x^2 - 1}{\sqrt{3x+4} - \sqrt{x+2}}$ $x^2 - 1 = (x-1)(x+1)$

$$\frac{(x-1)(x+1)(\sqrt{3x+4} + \sqrt{x+2})}{(3x+4) - (x+2)} = \frac{(x-1)(x+1)(\sqrt{3x+4} + \sqrt{x+2})}{2x+2}$$

$$= \frac{(x-1)(\sqrt{3x+4} + \sqrt{x+2})}{2}$$

$$\frac{(-2)(1+1)}{2} = -2$$

b. $\lim_{x \rightarrow 2} \frac{\sqrt{x+1} - \sqrt{9-3x}}{x-2}$ $= \frac{(x+1) - (9-3x)}{(x-2)(\sqrt{x+1} + \sqrt{9-3x})}$

$$= \frac{4x-8}{(x-2)(\sqrt{x+1} + \sqrt{9-3x})}$$

$$= \frac{4(x-2)}{(x-2)(\sqrt{x+1} + \sqrt{9-3x})}$$

$$= \frac{4}{\sqrt{2+1} + \sqrt{9-3(2)}}$$

$$= \frac{4}{\sqrt{3} + \sqrt{3}}$$

$$= \frac{4}{2\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}}$$

Tentukan bahwa $a^2 + 3b^2 = 8$

$$(-2)^2 + 3\left(\frac{2}{\sqrt{3}}\right)^2$$

$$= 4 + 3 \cdot \frac{4}{3}$$

$$= 4 + 4 = 8$$

Determine

Figure 1. FI-1's written solution showing complete coordination of both limit results

FI-1's written work showed a complete Action → Process → Object → Schema pathway (see Figure 1). The interview confirmed that the transformations were internally coordinated rather than reproduced only from an external cue. Scaffolding therefore served a consolidating and fading role: it elicited justification but did not change the solution pathway. Abstraction, conceptual reorganization, and formal reasoning were explicit. Further support should shift toward generalization or alternative proof rather than repeat procedural guidance.

FI-2 independently chose factorization and conjugate rationalization and obtained $a = -2$ and $b = 2/\sqrt{3}$ (see Figure 2). However, several intermediate transformations remained implicit, and the initial written response stopped before verifying $a^2 + 3b^2 = 8$. In the stimulated-recall interview, FI-2 explained that attention had remained on finding the two limit values and that the final instruction had been overlooked. A Schema-oriented prompt asked how the completed values could be used together. The student then substituted both values and produced $4 + 3(4/3) = 8$ without further procedural guidance.

FI-2 stated, "I focused on finding a and b and thought the problem was finished. I did not notice the last instruction." After the Schema-oriented prompt, the student continued, "Now I can use both values: $(-2)^2 + 3(2/\sqrt{3})^2 = 4 + 4 = 8$." The response located the initial weakness in task integration and showed that a single relational prompt was sufficient to activate the missing Schema connection.

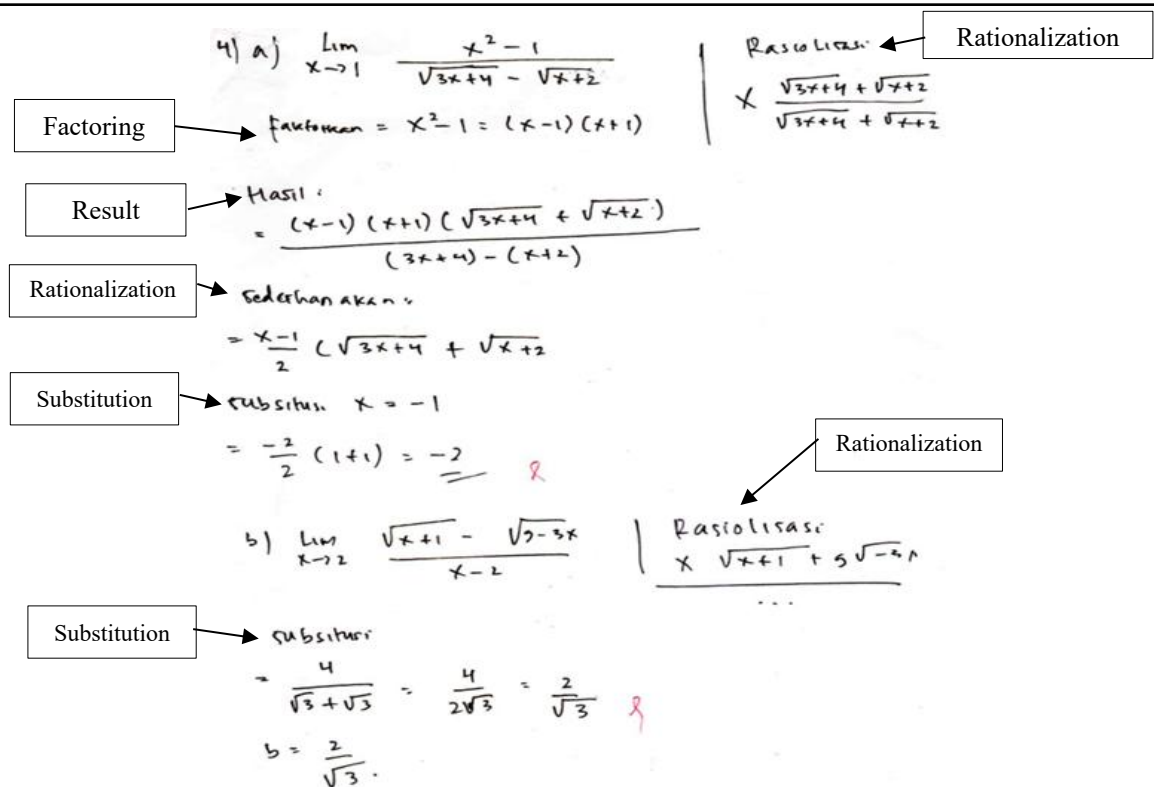


Figure 2. FI-2's written solution showing autonomous strategy selection but incomplete integration

FI-2 demonstrates that field independence supported autonomous strategy selection but did not initially guarantee complete integration. Action and Process constructions were stable, and valid limit values had become Objects. The Schema-oriented prompt activated the relation between those Objects and the required identity, producing an assisted shift from disconnected Objects to an integrated Schema. Abstraction and conceptual reorganization became explicit in the interview response, while formal reasoning was strengthened by making the previously omitted final link visible.

3.1.2. Field-Dependent Cases: Procedural Recognition Without Stable Coordination

FD-1 recognized both factorization and conjugate rationalization (see Figure 3). In the first limit, however, the student substituted before rewriting $2x+2$ as $2(x+1)$ and canceling the common factor, obtaining $a = 0$ rather than $a = -2$. The second limit was completed correctly, giving $b = 2/\sqrt{3}$, but the invalid value of a produced $a^2 + 3b^2 = 4$. During the interview, Process-oriented prompts asked the student to simplify the rationalized denominator, identify the common factor, and decide whether substitution should occur before or after cancellation. FD-1 then corrected a to -2 and verified the final value 8.

FD-1 reflected, "I substituted too early. I did not see that $2x + 2$ could be written as $2(x + 1)$." After the lecturer asked the student to identify the common factor and reconsider the order of operations, FD-1 concluded, "The factor $x + 1$ must be cancelled before substitution; then $a = -2$." The interview therefore identified unstable sequencing, rather than absence of the relevant algebraic Actions, as the source of the error.

$$\begin{aligned}
 a. \quad & \lim_{x \rightarrow -1} \frac{x^2 - 1}{\sqrt{x+1} - \sqrt{x+2}} \\
 & = x^2 - 1 = (x-1)(x+1) \\
 & = x \frac{\sqrt{x+1} + \sqrt{x+2}}{\sqrt{x+1} + \sqrt{x+2}} \\
 & = -(x-1)(x+1)(\sqrt{x+1} + \sqrt{x+2}) \\
 \text{Sub } x &= (-1, 0, \dots) \rightarrow 0 \quad \times \\
 b. \quad & \lim_{x \rightarrow 2} \frac{\sqrt{x+1} - \sqrt{9-3x}}{x-2} \\
 \text{Sub } & \frac{\sqrt{3} - \sqrt{3}}{0} = \frac{0}{0} \\
 & \times \frac{\sqrt{x+1} + \sqrt{9-3x}}{\sqrt{x+1} + \sqrt{9-3x}} \\
 & = \frac{(x+1) - (9-3x)}{(x-2)(\sqrt{x+1} + \sqrt{9-3x})} = \frac{4x-8}{(x-2)(\dots)} = \frac{4(x-2)}{(x-2)(\dots)} \\
 \text{Sub } x &= 2 \quad \frac{4}{\sqrt{x+1} + \sqrt{9-3x}} = \frac{4}{2\sqrt{3}} = \frac{2}{\sqrt{3}} \quad \checkmark \\
 \therefore a^2 + 3b^2 & \\
 a=0, \quad b &= \frac{2\sqrt{3}}{3} \\
 b^2 &= \frac{4 \cdot 3}{9} = \frac{4}{3} \\
 a^2 + 3b^2 &= 0 + 3 \cdot \frac{4}{3} = 4 \quad \times
 \end{aligned}$$

Figure 3. FD-1's written solution showing partial procedural control and unstable coordination

FD-1's initial difficulty occurred at Process coordination, not at recognition of the relevant Actions. The incorrect value of a had been prematurely encapsulated as an Object and carried into the final relation. Process-oriented scaffolding stabilized the transformation sequence and enabled construction of a valid Object, which was then integrated into the Schema. The resulting AMT development was assisted rather than fully independent: abstraction was already emerging, while conceptual reorganization and formal reasoning became more coherent after the lecturer's sequencing and validation prompts.

FD-2 remained at the most surface-level initial pathway (see Figure 4). The student used direct substitution in both limits, recorded $a = 0$ in the first, and stopped at $0/0$ in the second. In the interview, the student acknowledged not knowing that $0/0$ required algebraic reorganization. Action-oriented prompts first asked whether $x^2 - 1$ could be factored and what conjugate corresponded to each radical difference. With further step-by-step questions, FD-2 identified the common factors, obtained $a = -2$ and $b = 2/\sqrt{3}$, and completed the final identity.

FD-2 stated, "I only thought of direct substitution. When I obtained zero over zero, I did not know what to do next." Following explicit Action cues, the student said, "After being reminded, I remembered that $x^2 - 1 = (x - 1)(x + 1)$ and that a difference of radicals needs its conjugate." The student also acknowledged, "I can follow the steps after guidance, but I still cannot choose the next step independently." These excerpts distinguish scaffolded performance from independent construction.

Handwritten mathematical work by student FD-2. The work shows two limit problems, (a) and (b), with the student's attempt at direct substitution. Problem (a) is $\lim_{x \rightarrow -1} \frac{x^2-1}{\sqrt{3x+4}-\sqrt{x+2}}$ and problem (b) is $\lim_{x \rightarrow 2} \frac{\sqrt{x+1}-\sqrt{9-3x}}{x-2}$. The student's calculations for both result in $\frac{0}{0}$. Annotations include "Given", "Show that", and "and".

Figure 4. FD-2's written solution stopping at direct substitution and the indeterminate form 0/0

FD-2 initially remained at direct-substitution Action, with abstraction, conceptual reorganization, and formal reasoning not evident in the written response. Explicit Action-oriented cues enabled guided factorization and conjugate rationalization, followed by an emerging Process as the student followed the sequence of simplification and substitution. The response demonstrates scaffolded emergence, not independent mastery: FD-2 still required external organization to select and coordinate the transformations. Object and Schema use became visible only under close guidance.

3.1.3. Cross-Case Comparison of Cognitive Style, APOS Construction, and AMT Evidence

The strongest FI-FD difference appeared in initial task entry and the amount of external organization used. Both FI cases independently selected relevant transformations, while the FD cases relied more heavily on surface procedures or lecturer prompts. However, the developmental evidence was not reducible to the FI-FD binary. FI-2 needed a Schema prompt despite autonomous strategy selection; FD-1 needed Process support to stabilize known procedures; and FD-2 needed explicit Action cues before higher constructions could emerge. The cases therefore differed in the location, depth, and independence of change.

Table 4. Cross-case diagnostic comparison of cognitive style, APOS, and AMT

Case	Initial task organization	Initial dominant APOS pattern	Initial AMT evidence	Scaffold indicated
FI-1	Autonomous and complete restructuring	Action → Process → Object → Schema	All dimensions explicit	Fade cues; request justification or generalization.
FI-2	Autonomous strategy selection; incomplete integration	Stable Process; valid Objects; incomplete Schema	All dimensions emerging	Schema-oriented integration prompt.
FD-1	Relevant procedures; premature substitution	Actions available; Process unstable; invalid Object	All dimensions emerging	Process sequencing and validation prompts.
FD-2	Direct substitution; stopped at 0/0	Direct-substitution Action only	All dimensions not evident	Action cue, then stepwise Process support.

Table 4 diagnoses the APOS breakpoint visible in each initial solution. A cognitive-style label alone did not identify one uniform instructional need. The scaffold had to begin at

the construction that was not yet stable and then be reduced when the student continued independently. The following comparison makes the observed prompt-response sequence explicit.

3.1.4. Developmental Responses to APOS-Based Scaffolding

Table 5 separates each initial APOS construction from the student's supported response.

Table 5. Within-case developmental responses to APOS-based scaffolding

Case	Initial APOS position	Stage-specific scaffold	Observed response and interpretation
FI-1	Complete Schema	Fading justification prompt	Justified cancellation independently; support consolidated explicit AMT.
FI-2	Valid Objects; incomplete Schema	Schema integration prompt	Connected a and b to the identity; assisted Object-to-Schema shift.
FD-1	Unstable Process; invalid Object	Sequencing and validation prompts	Corrected the sequence, a, and the proof; Process stabilized.
FD-2	Direct-substitution Action	Action cues and stepwise Process prompts	Resolved 0/0 under close guidance; guided Action and emerging Process.

Across cases, scaffolding consolidated FI-1's stable reasoning, activated FI-2's missing Schema connection, repaired FD-1's unstable Process, and made a viable algebraic Action visible to FD-2. Cognitive style influenced reliance on external structure, while the APOS breakpoint identified the function of support.

3.1.5. Classroom Observation Evidence and Implementation Fidelity

Classroom observation records supplied a separate source of evidence for the enactment of APOS-based scaffolding. Table 6 summarizes representative episodes from the four sessions and shows how the planned prompts were translated into visible classroom activity. The observations were used to verify implementation and contextualize the case data; they were not used to assign cognitive-style categories or replace the mathematical evidence in students' written work.

Table 6. Representative classroom observation evidence and implementation fidelity

Observation code	APOS focus	Observed lecturer action	Observed student response	Fidelity interpretation
OBS-S1	Action	The lecturer asked why direct substitution produced 0/0 and directed attention to factorization and conjugate forms without supplying the completed transformation.	Some students selected a valid transformation immediately, whereas others repeated substitution or waited for an explicit first-step cue.	Action-oriented scaffolding was enacted. The variation in response showed different levels of access to the initial algebraic Action.

Observation code	APOS focus	Observed lecturer action	Observed student response	Fidelity interpretation
OBS-S2	Process	Students were asked to order the transformations, explain why cancellation was valid, and predict the simplified expression before substitution.	Students who knew the relevant procedures differed in coordination. One focal pathway omitted inferential links, while another used substitution before all common factors had been exposed.	Process-oriented prompts made sequencing and justification visible and identified unstable coordination rather than lack of procedural knowledge.
OBS-S3	Object	The lecturer required students to check the completed values of a and b against the simplified expressions before reusing them.	Some students validated both values independently. Others initially accepted an incorrect intermediate result until a checking prompt redirected attention to the preceding transformation.	Object-oriented scaffolding was enacted by requiring validation before a limit value was treated as a reusable entity.
OBS-S4	Schema	The lecturer asked students to connect both limit values and construct a complete verification of $a^2 + 3b^2 = 8$, then reduced support when students continued independently.	Some students completed the integration without further assistance; others stopped after finding a and b or required stepwise prompts to connect the two results.	Schema-oriented activity and scaffold fading were documented. The observation distinguished complete integration from correct but disconnected results.

The observation records confirmed that all four APOS orientations were enacted and that support was not applied uniformly. Action prompts made a viable transformation visible, Process prompts stabilized the order and meaning of transformations, Object prompts required validation of completed values, and Schema prompts connected the two limit results. These records strengthen the empirical basis of the study because the implementation claim is supported by documented classroom episodes in addition to written work and interview responses.

3.2. Discussion

The central finding is not that field-independent students possess stronger advanced mathematical thinking than field-dependent students. Rather, cognitive style shaped how students initially extracted, organized, and acted on information, while the location of the mathematical breakdown determined the support that became useful. Variation within each category is theoretically important. Autonomous selection of factorization or conjugate rationalization did not always produce a complete argument, and reliance on external structure did not imply an absence of relevant mathematical knowledge. These contrasting cases challenge a simple FI-success and FD-difficulty interpretation and support the use of field dependence-independence as an analytical lens rather than a hierarchy of mathematical ability (Hattie & O’Leary, 2025; Witkin et al., 1977).

This interpretation refines the role assigned to cognitive style in scaffolded calculus learning. Field independence can facilitate early disembedding of algebraic structures, whereas field dependence can increase the usefulness of visible organization and focused instructional cues. However, cognitive style alone does not identify what a student understands or which prompt should be delivered. The same learner category may contain students who differ in procedural availability, coordination, validation, and integration. This finding is consistent with evidence that scaffolding effects depend on the resources already available to the learner and on the specific mechanism targeted by the activity (Hofer & Reinhold, 2025). The relevant unit for instructional diagnosis is therefore the student's current mathematical construction, not the cognitive-style label.

APOS theory clarifies the mechanism through which different prompts supported reasoning. The four forms of support performed distinct epistemic functions rather than representing increasing amounts of help. An Action-oriented cue made a viable transformation visible. A Process-oriented prompt coordinated the order and meaning of transformations. An Object-oriented question required the student to validate a completed limit value before reusing it. A Schema-oriented prompt connected separate limit results within one coherent argument. These constructions were not necessarily formed in a uniform or strictly linear sequence. A student could execute appropriate Actions without coordinating them, or could encapsulate an invalid result as an Object after an unstable Process. This diagnostic use of APOS extends its value beyond classifying solution steps and supports decisions about when scaffolding should be redirected, intensified, or faded (Arnon et al., 2014; Borji et al., 2022; Cottrill et al., 1996; Listiawati et al., 2025; Martínez-Planell et al., 2024).

The AMT framework explains why procedural correctness alone was insufficient evidence of advanced reasoning. In the limit task, abstraction became visible when students recognized and reused the difference-of-squares or conjugate structure. Conceptual reorganization appeared when separate symbolic transformations and intermediate values were coordinated into a valid solution. Formal reasoning required students to justify cancellation, distinguish an indeterminate form from a limit value, and connect the two limits to the final identity. This interpretation is consistent with accounts of limit understanding that emphasize coordinated processes, defensible concept structures, and meaningful relations rather than isolated routines (Cottrill et al., 1996; Santosa & Filiz, 2025; Tall, 1991; Tall & Vinner, 1981; Thompson & Harel, 2021). It also addresses persistent findings that prospective teachers may obtain results without constructing stable meanings for limits (Bansilal & Mkhwanazi, 2022; Mirin, 2024).

A further implication is that similar final answers can represent different developmental mechanisms. A student who already recognizes the relevant algebraic structure may need support only to make inferential links explicit or integrate completed values. Another student may know the required procedures but need help sequencing them. A third may require an initial structural cue before any viable transformation becomes available. Correctness should therefore be interpreted together with the origin of the strategy, the quality of the explanation, and the student's ability to continue after support is reduced. This distinction prevents a correct answer produced through close guidance from being treated as equivalent to an independently constructed and justified solution.

The term developing is supported here through evidence of short-term qualitative reorganization during scaffolded activity. Development was inferred when a prompt enabled a student to notice a previously unused relation, coordinate an incomplete sequence, validate an intermediate result, or integrate separate results, and when the student subsequently continued with less assistance. By contrast, following a worked cue without being able to explain or reuse the transformation was interpreted as guided performance rather than a stable construction. This mechanism-based interpretation is compatible with conceptual scaffolding research (Kusmaryono et al., 2020; Setiyani et al., 2025), but it does not imply a statistically measured treatment effect or durable learning gain (Chien et al., 2026; DiNapoli & Miller, 2022; Hofer & Reinhold, 2025).

The findings suggest a practical diagnostic cycle for function-limit instruction. First, the lecturer examines where the student's written reasoning stops or becomes invalid. Second, the lecturer selects the least directive prompt that targets the missing construction. A student who stops at 0/0 may need an Action cue that draws attention to factorization or a conjugate. A student who knows both procedures but applies them in an invalid order needs a Process prompt. Correct values accepted without checking call for Object-oriented validation, while separate results that are not connected require Schema-oriented integration. Third, the lecturer checks whether the student can explain and continue independently, then reduces support. This approach avoids both over-scaffolding students who can reorganize the task autonomously and under-supporting students whose initial Actions have not stabilized.

The study contributes a three-layer account of scaffolded AMT development. Cognitive style explains variation in initial task organization and access to external structure. APOS locates the construction that is available, unstable, or missing, while AMT describes the quality of abstraction, conceptual reorganization, and justification that emerges. Methodologically, triangulation of written work, interviews, and classroom observations keeps mathematical evidence central and retains disconfirming cases. The contribution is therefore not a typology prescribing one method for FI students and another for FD students, but a diagnostic model linking learner variation to the mathematical function of scaffolding.

Several limitations bound this interpretation. The study involved one institution, one integrated task, and four purposively selected cases, so it supports analytical transfer rather than statistical generalization. The observed changes were immediate and cannot establish retention, transfer, or independent long-term development. Because stimulated-recall interviews may extend reasoning beyond the initial written work, classroom learning and interview-elicited change must remain analytically distinct. GEFT categories were used for case selection, not as fixed diagnoses or measures of mathematical ability. Future studies should use repeated novel tasks, delayed assessments, systematic records of scaffold fading, and multi-site longitudinal designs.

4. CONCLUSION

This qualitative case study found that FI and FD students followed distinct but nonuniform developmental pathways on the same integrated function-limit task. FI-1 entered with a complete Schema and used prompts mainly to articulate justification. FI-2 integrated

correct limit Objects after a Schema-oriented prompt. FD-1 corrected an invalid result after Process-oriented prompts stabilized the transformation sequence. FD-2 moved beyond direct substitution only after explicit Action cues and close coordination support.

Combining cognitive style, APOS, and AMT explains both who used external structure and what the support needed to accomplish. Cognitive style clarified task entry, APOS located the unstable construction, and AMT described the qualitative change in abstraction, conceptual reorganization, and formal reasoning. The role of APOS-based scaffolding was stage-specific: Action cues supported strategy selection, Process prompts stabilized transformations, Object questions validated completed values, and Schema prompts integrated separate results.

The study supports stage-specific, evidence-based scaffolding rather than fixed prescriptions for FI or FD categories. Future work should examine whether these supported constructions persist, transfer to new limit tasks, and become independent across longer learning trajectories.

This study was conducted at one institution and involved only four purposively selected students from a class of 24 participants. Therefore, the findings support analytical transfer rather than statistical generalization. The study also used one integrated function-limit task, so the identified APOS constructions may differ when students solve graphical, numerical, formal, or unfamiliar limit problems. In addition, the observed changes occurred during four instructional sessions and stimulated-recall interviews. The study could not confirm whether the supported constructions were retained, transferred to new problems, or performed independently after scaffolding was removed. The interview prompts may also have extended students' reasoning beyond their initial written responses. Furthermore, GEFT classifications were used only for case selection and should not be interpreted as fixed diagnoses, measures of mathematical ability, or direct predictors of learning success.

Future studies should involve larger and more diverse samples across several institutions to examine whether the identified patterns remain consistent in different educational contexts. Researchers should also use repeated and varied tasks, including graphical limits, one-sided limits, infinite limits, continuity, and formal limit definitions, to investigate transfer across mathematical representations. Longitudinal designs with delayed assessments are needed to determine whether scaffolded Actions, Processes, Objects, and Schemas persist after support has been withdrawn. Future research should systematically record the number, sequence, intensity, and fading of prompts to distinguish guided performance from independent development. Mixed-methods, quasi-experimental, or design-based studies may also test learning gains while retaining detailed qualitative evidence of students' reasoning. Finally, researchers should analyze cognitive style as a flexible tendency rather than a fixed category and strengthen interpretive reliability through multiple coders, participant validation, and systematic analysis of interview translations.

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