Infinity Journal of Mathematics Education
Volume 15, No. 4, 2026
https://doi.org/10.22460/infinity.v15i4.p1271-1300

p-ISSN 2089-6867
e–ISSN 2460-9285

# Networking ETC, APOS and OSA to analyze students' understanding of the derivative in a graphical context

**Camilo Andrés Rodríguez-Nieto[1], Claudio Fuentealba[2*], Vicenç Font Moll[3], Flor Monserrat Rodríguez-Vásquez[4], Andrea Cárcamo[2]**

[1]Department of Natural and Exact Sciences, Universidad de la Costa, Barranquilla, Colombia
[2]Centro de Docencia de Ciencias Básicas para Ingeniería, Universidad Austral de Chile, Valdivia, Chile
[3]Departament de Didàctica de les Ciències Experimentals i la Matemàtica, Universitat de Barcelona, Barcelona, Spain
[4]Facultad de Matemáticas, Universidad Autónoma de Guerrero, Chilpancingo, México
[*]Correspondence: cfuentealba@uach.cl

**Received**: Jun 8, 2026 | **Revised**: Nov 13, 2026 | **Accepted**: Nov 14, 2026 | **Published Online**: Nov 25, 2026

## Abstract

This study analyzes how six first-year mathematics education students at a Mexican university understand the derivative when they reconstruct a possible graph of a function from the graph of its derivative, working on one graphical task and a task-based interview. We articulate the Extended Theory of Connections (ETC), the Action-Process-Object-Schema (APOS) theory and the Onto-Semiotic Approach (OSA) through a local networking of theories, asking what the three reveal that none accounts for alone. Two researchers coded the material independently as mathematical practices, primary objects and semiotic functions, then as actions, processes, objects and schemas, and finally as mathematical connections. In this group, four students coordinated implication, representation, reversibility, procedural and part-whole connections, two of them meaning and feature as well, rebuilt the global behavior of the function and reached a trans level. Partial coordination, with inconsistencies in concavity and inflection points, corresponded to an inter level, and reading the graph of the function as a mirror of its derivative to an intra level. The study offers a local analytical model in which the semiotic function supplies the criterion that decides which connection is at stake and the APOS level names how far those connections have been coordinated.

**Keywords**:
APOS theory, Derivative, Mathematical connections, Networking of theories, Onto-semiotic approach

**How to Cite**:
Rodríguez-Nieto, C. A., Fuentealba, C., Font Moll, V., Rodríguez-Vásquez, F. M., & Cárcamo, A. (2026). Networking ETC, APOS and OSA to analyze students' understanding of the derivative in a graphical context. *Infinity Journal*, *15*(4), 1271-1300. https://doi.org/10.22460/infinity.v15i4.p1271-1300

*This is an open access article under the CC BY-SA license.*

## 1. INTRODUCTION

The literature in Mathematics Education suggests that understanding a mathematical concept involves using it competently in problem solving and establishing connections among representations, meanings, and everyday contexts (Hiebert & Carpenter, 1992; Rodríguez-Nieto & Font Moll, 2025). The NCTM (2000) adds that understanding involves the appropriate

use of mathematical objects and the relating of concepts and procedures across different contexts. Along these lines, Cai and Ding (2017) noted that some teachers conceive of understanding as a network of connections built through the continuous articulation of different mathematical ideas. In the particular case of the derivative, this view implies that understanding is visible when a student relates the sign and the variation of one function to the shape of another, rather than when the student merely applies a rule of differentiation.

Mathematical connections have been studied from a variety of perspectives and theoretical articulations. The ETC has been integrated with the OSA to analyze the practices, objects, and semiotic functions involved in mathematical understanding (Rodríguez-Nieto, Font Moll, et al., 2022), and with the theory of conceptual metaphor to explain the construction of meanings through metaphorical expressions (Rodríguez-Nieto, Rodríguez-Vásquez, et al., 2022). With regard to the derivative, most studies have been carried out with pre-university students (Dolores-Flores & García-García, 2017; García-García & Dolores-Flores, 2018, 2021), reporting meaning, feature, procedural, different-representations, modelling, and part-whole connections. These studies describe which connections students establish, yet they do not explain what makes a given fragment of activity count as one connection rather than another, nor how the connections a student establishes relate to the cognitive structure behind them. That double omission is the gap this study addresses.

Other research has emphasized the arguments constructed by pre-service teachers about the derivative (Vargas González et al., 2020) and the difficulties in understanding this concept in graphical contexts. Several studies provide evidence that students experience difficulties coordinating different representations of the derivative (Ariza & Llinares Ciscar, 2009; Galindo-Illanes et al., 2025; Pino-Fan et al., 2018; Pino-Fan et al., 2017; Sánchez-Matamoros et al., 2015). Difficulties in establishing bidirectional relationships among $f$, $f'$ and $f''$, associated with reversibility connections, have also been reported (Fuentealba, Badillo, & Sánchez-Matamoros, 2019). García-García and Dolores-Flores (2021) argue that many students rely on explicit algebraic expressions to graph derivatives or antiderivatives. Likewise, Calculus students can differentiate functions but show difficulties in conceptualizing these processes in non-algebraic contexts (Voskoglou, 2017).

The study of graphical contexts of the derivative is pertinent because of the difficulties students face in establishing mathematical connections and in understanding its formal definition as a limit (Fuentealba, Badillo, & Sánchez-Matamoros, 2019; Rodríguez-Nieto et al., 2025). Such understanding requires coordinating prior meanings, such as function, continuity, and limit, with symbolic representations that are complex for novice students. Textbook analyses add that the interpretations of the derivative are arranged with uneven attention to the graphical connection between the function and its derivative (Guo & Yang, 2026). Accordingly, promoting multiple representations of the derivative is recommended (Voskoglou, 2017). Berry and Nyman (2003) point out that connecting the graph of a function with that of its derivative, and vice versa, strengthens conceptual understanding of Calculus. Likewise, Rodríguez-Nieto et al. (2023) and Rodríguez-Nieto and Font Moll (2025) underline the relevance of reversible processes between $f$ and $f'$. Nevertheless, difficulties persist, such as interpreting $f$ as a "mirror" of $f'$ without considering the slope of the tangent line (Rodríguez-Nieto et al., 2025).

Recent research highlights the ongoing relevance of APOS theory and the need to update it. Nahari et al. (2025) emphasize its conceptual robustness and its adaptability to contemporary contexts. Syaiful et al. (2025) and Lastari et al. (2025) show its effectiveness for the development of critical thinking and procedural fluency, while Ramadiana et al. (2026) highlight its connection with relational understanding. Matindike (2025) shows, with Autograph, that a dynamic tool can support the encapsulation of a function as an object and the flexible passage between its algebraic and graphical representations. Romero Osorio et al. (2026) report that understanding of the definite integral remains fragmented across its algebraic, geometric and conceptual representations until the process is encapsulated as an object. Borji et al. (2018), from the OSA–APOS framework, point out limitations in the coordination of signs, Processes, Objects, and Schemas. These contributions share a limitation. APOS names the cognitive structure but does not say which observable fragment of activity signals it, the ETC names the connection but leaves the criterion that identifies it implicit, and the OSA describes the activity in fine detail yet does not grade how far it has been integrated. Each framework is therefore short of exactly what another supplies. Articulating the three is not an addition of lenses but a way of closing that circle, because the semiotic function of the OSA provides the criterion that decides which ETC connection is at stake, and the APOS levels grade how far those connections have been coordinated. Accordingly, this study addresses two questions. First, which mathematical connections do university students establish when they reconstruct the graph of a function from the graph of its derivative, and through which semiotic functions are those connections built. Second, how does the coordination of those connections relate to the intra, inter and trans levels of development of the derivative schema.

Putting theories into dialogue is an established practice in Mathematics Education research, and it calls for attention to where each framework comes from and to what it is built to do (Bikner-Ahsbahs & Johnson, 2026; Scheiner, 2026). Three views of understanding are at stake here, one for each of the frameworks articulated. From the ETC, to understand is to establish multiple connections between representations, meanings and contexts, and the linkage of those connections is itself the indicator of understanding (Rodríguez-Nieto, Font Moll, et al., 2022; Rodríguez-Nieto, Rodríguez-Vásquez, et al., 2022). In APOS theory, understanding develops through a schema, so that high levels of understanding of the derivative call for tasks that demand the use of its meanings across different modes of representation and the conversion among them (Badillo et al., 2011; Fuentealba et al., 2015; García et al., 2011).

In the OSA, articulated with the ETC, understanding is the cognitive process by which a subject, in their practice, relates primary objects (situations, concepts, propositions, representations, arguments, procedures) to one another and to other domains through semiotic functions, grouped as ETC connections. Understanding a concept enables the subject to use it competently (Godino et al., 2019; Rodríguez-Nieto et al., 2025).

### 1.1. Extended Theory of Connections (ETC)

In this study, a mathematical connection is understood metaphorically, from the articulation between the ETC and the OSA, as the tip of an iceberg made up of a cluster of practices, processes, and primary objects that emerge in a subject's mathematical activity when

solving a task (intra- or extra-mathematical), together with the semiotic functions that relate them (Rodríguez-Nieto, Rodríguez-Vásquez, et al., 2022). Because of the type of task, we consider only intra-mathematical connections (Businskas, 2008; García-García & Dolores-Flores, 2021; Rodríguez-Nieto, Font Moll, et al., 2022), as described below.

1) Procedural: appears when the subject uses rules, algorithms, or formulas to complete or solve a mathematical task. These connections have the form: A is a procedure used to work with concept B (Businskas, 2008).
2) Different representations: identified when the subject represents a mathematical concept using alternative or equivalent representations (Businskas, 2008). Equivalent representations are transformations carried out within the same representation or register (algebraic–algebraic). Alternative representations refer to those in which the register in which they were formed is modified (graphical–symbolic).
3) Feature: identified when the subject expresses features of mathematical concepts or descriptions of their properties in terms of other concepts that make them different from, or similar to, other concepts (Eli et al., 2013; García-García & Dolores-Flores, 2021).
4) Reversibility: occurs when a subject starts from a concept A to reach a concept B and reverses the process, starting from B to return to A (García-García & Dolores-Flores, 2021).
5) Part-whole: this type of connection appears when the subject establishes logical relationships in two ways (general-particular and inclusion). The generalization relationship has the form: A is a generalization of B, and B is a particular case of A (Businskas, 2008).
6) Meaning: identified when a student attributes a meaning to a mathematical concept, that is, what it means to them. It includes cases in which a student gives a definition they have constructed for these concepts (García-García & Dolores-Flores, 2021). It is also recognized when meaning is used to solve a problem.
7) Implication: identified when a concept P leads to another concept Q through a logical relation (P→Q) (Businskas, 2008).
8) The remaining categories of the ETC are not activated by this task and are therefore only named here. The metaphorical connection projects properties of a known domain onto a less known one (Rodríguez-Nieto, Font Moll, et al., 2022), the metaphorical connection based on mnemonics links a mnemonic rule to a mathematical object or procedure (Rodríguez-Nieto et al., 2024), and the idealising connection dematerializes an ostensive object and turns it into an ideal one (Ledezma et al., 2024).

### 1.2. Action-Process-Object-Schema (APOS)

APOS theory (Action, Process, Object, and Schema) follows a constructivist approach based on the reflective abstraction introduced by Piaget for the construction of mathematical knowledge (Arnon et al., 2014; Dubinsky & Mcdonald, 2001). In this theory, a concept is conceived from Actions, Processes, and Objects organized into Schemas.

An Action is an externally directed transformation of one or more previously conceived Objects. When Actions are internalized, they become Processes. That is to say, the subject is aware of and has command over the transformation caused by the Action in an

anticipated manner and without external stimulation. The change of structure from Action to Process is called interiorization (repetition and reflection on Actions). New Processes can also be built from Processes already constructed through the mechanisms of coordination and reversal (Arnon et al., 2014). Likewise, when the subject conceives of the Process as a totality on which Actions can be performed, the Process is said to have been encapsulated into an Object, and if the subject returns to the Process that gave rise to the Object, they have performed a mechanism of de-encapsulation (Asiala et al., 1996).

Finally, a Schema of a mathematical concept is understood as the collection of Actions, Processes, Objects, and other Schemas (if applicable) and their connections, which the subject uses to solve problems related to the concept (Arnon et al., 2014). It develops through three levels: intra, inter, and trans. Sánchez-Matamoros (2004) characterizes these levels by the logical relations the student establishes and by the use made of the modes of representation. At the intra level the elements are used in isolation and any attempt to relate them carries errors. At the inter level logical relations appear, but only among elements expressed in the same mode of representation. At the trans level the repertoire of relations widens and the modes of representation are synthesized, which is what allows the mathematical structure to be built.

In APOS, Objects emerge mainly through two mechanisms. First, encapsulation, based on reflective abstraction, consists of conceiving a Process as a complete whole so as to characterize it and study its properties, regarding Objects as abstract notions with properties and representations. Second, thematization implies thinking of the Schema as a whole, analyzing its transformations and studying the relationships among its parts (Borji et al., 2018; Fuentealba et al., 2018). APOS theory also emphasizes the importance of the genetic decomposition, understood as a theoretical analysis of the Actions, Processes, Objects, and Schemas that a subject may construct in order to learn a mathematical concept.

### 1.3. Onto-Semiotic Approach (OSA)

The OSA describes mathematical activity (both institutional and personal) by attending to the objects involved and the semiotic functions between them (Font et al., 2013), through a set of theoretical tools. The primary objects in the OSA are: language, problem situations, concepts/definitions, propositions, procedures, and arguments (Godino et al., 2019). Language encompasses terms and graphical representations in different registers, problem situations are tasks and applications, concepts are objects such as the derivative or the slope, propositions are statements about them, procedures are algorithms, and arguments are their justifications. These objects emerge through primary processes such as communication, definition, procedures, and argumentation (Godino et al., 2019). Furthermore, processes derived from the process–product dualities also intervene: personalization–institutionalization, materialization–idealization, representation–meaning, synthesis–analysis, and generalization–particularization (Font et al., 2013). Another central element of the OSA is the semiotic function (SF), understood as a triadic relation between an antecedent (expression/object) and a consequent (content/object), established by a subject through a criterion or correspondence code, which makes it possible to relate mathematical practices and objects (Godino et al., 2019).

These primary objects are articulated in cognitive and epistemic configurations, depending on whether the meanings at stake are personal or institutional, and it is the semiotic function that relates them to one another (Godino et al., 2019).

### 1.4. Relation Between APOS-OSA

APOS theory and the OSA were integrated in the networking by Font Moll et al. (2016), centered on the notion of object. Borji et al. (2018) describe differences and similarities that guide the use of the tools in the analysis (see Table 1).

**Table 1**. Combination and comparison of the theories

| Differences and similarities | Description |
|---|---|
| Action versus mathematical practice | The terms Action (APOS) and mathematical practice (OSA) are complementary and can be articulated coherently. A difference between action in APOS and practice in the OSA is that actions are mental constructions of an individual that transform mathematical objects through procedures, whereas practice in the OSA is a sequence of actions regulated by institutional or community norms. |
| Processes versus procedures | There is a close and significant relationship. For example, one of these objects in the OSA is the procedure, which is a rule of sequenced steps for carrying out the practice, or a part of the practice. A procedure can also be considered complementary to a process in APOS when an individual demonstrates having interiorized mental actions and is aware of the outcome of the process. |
| Encapsulation versus primary objects | The result of the encapsulation of a process is a mental object. In fact, encapsulation occurs when an individual applies an Action to a Process, regarding a dynamic structure (Process) as a static structure to which Actions can be applied. From the OSA, encapsulation can be described as a double change in nature: 1) from process to object, which is a primary object based on the OSA. 2) Encapsulation produces a change in the nature of the mathematical object associated with the cognitive object, for example, there is always a shift from a procedure to a definition. |
| Cognitive configuration versus schema | These terms are complementary. In APOS, a schema is a coherent set of actions, processes, objects, and other schemas related in the individual's mind, which enables them to decide how to solve a problem. The cognitive configuration of the OSA accounts for the objects involved in and emerging from personal practices, which makes it possible to describe the subject's knowledge, understanding, and skills during learning. Moreover, it reveals the complexity of a specific task associated with a schema. Although different, both constructs are complementary: the cognitive configuration is linked to a particular task, whereas the schema is related to types of tasks. |

Note: Information adapted from Borji et al. (2018)

The ETC was articulated with the OSA to improve the analysis of mathematical connections (Rodríguez-Nieto, Rodríguez-Vásquez, et al., 2022; Rodríguez-Nieto et al., 2023).

### 1.5. Relation Between ETC-OSA

Rodríguez-Nieto, Font Moll, et al. (2022) identified agreements and complementarities in the respective conceptions of mathematical connections in the ETC and the OSA. These theories complement each other to provide a more detailed analysis of mathematical connections and their functioning in mathematical activity. With the OSA tools, a connection is visualized as the tip of an iceberg: practices, primary objects, and the semiotic functions (SF) that relate them. In this networking, the data were first analyzed in terms of practices, configurations of primary objects, and SFs that relate to them as proposed by the OSA. Finally, parts of the mathematical activity (i.e., practices, primary objects, and SFs) were interpreted as a type of connection proposed in the ETC.

This networking between the ETC and the OSA contributes to research on "networking of theories" because it illustrates how the networking of two theories at different levels can be applied in a specific situation: a general theory for the analysis of mathematical activity, such as the OSA, and a specific framework for analyzing mathematical connections, such as the ETC.

The OSA allows us to identify the structure of mathematical activity that underlies the connection process. In this sense, it complements the ETC by providing a clear framework for analyzing mathematical activity and connections. For instance, the distinction between propositions and procedures facilitates the recognition of procedural and implication connections (Font et al., 2013). Moreover, in the ETC it is not always straightforward to determine the type of connection present in mathematical activity. The OSA, however, makes it possible to identify which mathematical objects are related through semiotic functions. On this basis, we argue that the OSA, the ETC, and APOS maintain relevant conceptual relationships and share the notion of connection as a central element for understanding mathematical activity.

## 2. METHOD

### 2.1. Local Coordination and Combination

This research follows a qualitative design (Cohen et al., 2013) and was developed in five stages. Stage 1: a theoretical reflection was carried out, framed within the third pair of strategies of the networking of theories between the OSA, APOS, and the ETC, with the aim of coordinating and combining them. Stage 2: in a reflection context, participants were purposively selected and collaborated voluntarily. Stage 3: a task from Spivak (1992) concerning a graphical context of the derivative was used in order to establish or promote connections. Stage 4: semi-structured task-based interviews were carried out for data collection, and in Stage 5 the data were analyzed by means of an integrative analysis method based on the tools of the OSA, APOS, and the ETC. Finally, the results are presented, showing how the mathematical connections established by the students relate to their understanding of the derivative concept.

Table 2 displays the functioning of the theoretical tools of the three theories so as to recognize understanding in terms of the intra, inter, and trans levels. It takes the comparison of Table 1 one step further, since it does not only contrast the tools but aligns them with the

development of the schema. Mathematical connections mediate between the OSA and APOS theories and, in turn, foster understanding.

**Table 2**. Coordination of OSA, APOS, ETC, and schema development

| OSA | APOS (Genetic Decomposition) | ETC | Levels (Schema Development) |
|---|---|---|---|
| Mathematical practices Observable activity of the subject, institutionally regulated and oriented towards problem solving. They constitute the empirical starting point of the analysis. | Actions Initial transformations that the subject performs on mathematical objects (explicit, operational and generally external). | Activation of connections Connections emerge as a result of practical activity; they represent initial relationships between mathematical elements. | Intra Isolated actions, without coordination; fragmented use of knowledge and of connections. |
| Processes Regularities in practices that reveal ways of doing mathematics (procedures, inferences, coordinations). | Processes Interiorization of actions; the subject is able to operate mentally and to coordinate transformations. | Coordination of connections Connections (implication, representations, meaning, etc.) begin to be articulated into more complex structures. | Inter Partial coordination between processes, representations and connections; relationships appear, but with inconsistencies. |
| Primary objects Mathematical entities that are mobilized (definitions, propositions, procedures, arguments, languages). | Objects Encapsulation of processes into stable cognitive entities (e.g. function, derivative, concavity). | Consolidation of connections Connections make it possible to give meaning to objects and to relate them to one another (e.g. part-whole, implication, feature). | Inter Objects begin to be organized, but they are not yet fully integrated into a coherent system. |
| Cognitive configuration Articulated system of primary objects that the subject mobilizes in solving a task. | Schema (structure) Coherent organization of actions, processes and objects that makes it possible to interpret and solve mathematical situations. | Network of mathematical connections Connections are integrated into a structured network that articulates meanings and relationships between objects. | Trans Coherent and global integration of knowledge; coordination of all the connections within a stable structure. |
| Semiotic functions (SF) Relationships between expressions and contents that account for the construction of meaning (graphical, symbolic, verbal registers, etc.). | Relationships within the schema Internal connections between actions, processes and objects mediated by representations. | Nature of the connections The SFs explain how connections are constructed (meaning, representation, reversibility, etc.). | Trans Flexible, conscious and coherent use of multiple representations and connections in different contexts. |

Source: Authors' own elaboration

In the earlier pairs of strategies for theoretical articulation, the theories were understood, compared, and contrasted. Then, they were coordinated and combined. Bikner-Ahsbahs and Prediger (2010) state that theoretical articulations are mainly aimed at a networked understanding of an empirical phenomenon, such as the episode on the understanding of the derivative in a graphical context analyzed here.

### 2.2. Context and Participants

The participants were six first-year students of the Bachelor of Mathematics with an emphasis on Mathematics Education at a university in southwestern Mexico: four men and two women, who took part voluntarily. All of them had taken and passed Differential Calculus, a course in which they studied functions, limits, derivatives and applications involving maxima and minima, monotonicity and optimization. With the authorization of the Dean of the Faculty of Mathematics, the first author carried out a non-participant observation during the eight classes on derivatives, from the definition to the applications. The aim was to identify the topics and subtopics addressed, as well as the way in which the instructor treated the graphical aspects of the derivative $f'$ and their relationship with the original function $f$. Henceforth, the participants are identified as P1, P2, P3, P4, P5, and P6. The six participants were selected purposively on three criteria fixed before the fieldwork. They had taken and passed Differential Calculus, so that the content was recent and comparable across the group. They had worked with the graphical register during that course, which the classroom observation confirmed. And they agreed to be interviewed individually and to have the session recorded. The group was self-selected among those who volunteered rather than stratified, and the participants were not chosen to represent contrasting levels of mathematical attainment. The levels of schema development reported below emerged from the analysis and were in no way a criterion of recruitment, so the study does not claim to cover a spectrum of ability.

The field notes from those eight sessions are not part of the analysed corpus and no result reported below rests on them. Their purpose was to establish what the students had been taught and in which registers, so that the task would not ask for anything outside their instruction, and they serve only to describe the context and to confirm that the graphical register had been worked on in class.

### 2.3. Ethical Procedures

The study followed the ethical procedures of the institution where the data were collected. The Dean of the Faculty of Mathematics authorized the classroom observation and the interviews. At the time the data were collected the university did not require review by an ethics committee for a study of this kind. Each participant signed an informed consent form stating the purpose of the study, the voluntary nature of participation, the right to withdraw at any moment without consequence, and the use of the material for research and publication. Consent for audio and video recording was requested separately and granted by all six participants. The recordings are kept on storage accessible only to the research team, and the transcriptions were anonymized at the moment of transcription by replacing names with the codes P1 to P6. The written productions reproduced in the figures carry no element that identifies their author. No personal data beyond what is reported here were collected.

## 2.4. Instruments and Data Collection Techniques

### *Task on the Derivative*

The task proposed to the students (see Figure 1) was selected because previous research has used it to study the understanding of the derivative from APOS theory (Fuentealba, Badillo, & Sánchez-Matamoros, 2019; Fuentealba, Badillo, Sánchez-Matamoros, et al., 2019; Fuentealba et al., 2015). Furthermore, it has been noted that solving this type of task fosters high levels of understanding of the derivative (Fuentealba et al., 2015). In this study we aim to identify the network of mathematical connections that enables solving the task and understanding the derivative in a graphical context. We thereby address the idea that establishing connections is an indicator of understanding (Rodríguez-Nieto et al., 2025). The task elicits the connections of interest because it withholds the algebraic expression. Having no formula to differentiate, the student cannot reach the answer by applying a rule and has to read the graph of $f'$ as information about $f$, which is where an implication connection becomes observable. Items a and b reverse the usual direction of work, from derivative to function, which is where a reversibility connection can appear. Items c and d require the second derivative to be read off the variation of the first, so that two levels of derivation have to be held together. And because the expected answer is a sketch rather than a number, the student has to assemble the intervals into a single curve, which is where a part-whole connection can be observed.

Task: the graph corresponds to the derivative $f'$ of the function f; sketch the possible graph(s) of the function $f$, and determine:

a) The intervals of increase and decrease of $f$.
b) The maximum or minimum values of $f$.
c) The intervals where $f$ is concave up or concave down.
d) The inflection points.

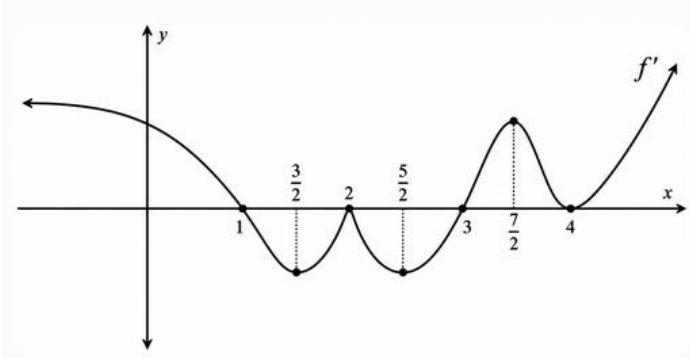



**Figure 1**. Graph of the derivative $f'$ (taken from Spivak (1992))

### *Task-based Interview*

A task-based semi-structured interview was used, following the perspective of Qu and Dumay (2011), in order to foster dialogue between researcher and student. The interview lasted approximately two hours and was conducted after the students had individually solved the proposed task (see Figure 1). The researcher then asked questions grounded in the response

protocol and in the participants' written work. This interview was suitable because it allows for broad, argued, and flexible answers (Qu & Dumay, 2011). Video recordings, audio, and field notes were used to capture verbal productions and actions.

### 2.5. Data Analysis

To analyze the data, we used the method for analyzing mathematical activity based on the tools of the OSA, APOS theory, and the ETC (see Figure 2). The procedure was applied to each participant in six steps. Step 1, the interview and the written production were transcribed in full and the transcript was segmented into units of activity, each unit being a stretch of talk or writing devoted to one item of the task. Step 2, each unit was described as a mathematical practice and assigned to one of the categories MP1 to MP10 of the a priori analysis. Step 3, the primary objects mobilized in the unit were listed using the six categories of the OSA. Step 4, the semiotic functions were written as ordered pairs of expression and content together with the correspondence code the student used, which is the criterion that fixes what the unit means. Step 5, the pair of objects related by each semiotic function was matched against the operational indicators of Table 5, which assigns the unit to a type of ETC connection. Step 6, the set of connections established by the participant across the whole task was read against the criteria of Table 6 to assign a level of development of the schema. Steps 2 to 5 are local and run unit by unit, whereas step 6 is global and uses the complete record of the participant.

Two of the authors coded the material independently, working from the same written definitions and without access to each other's coding. Disagreements were discussed against the transcript until agreement was reached, and the rule used to settle each one was written into the operational indicators of Table 5 so that it applied to the whole corpus and not only to the case that raised it. Credibility was further supported by triangulating three sources for every participant, the written production, the recording of the resolution and the interview, and no connection is reported unless it appeared in at least two of them.

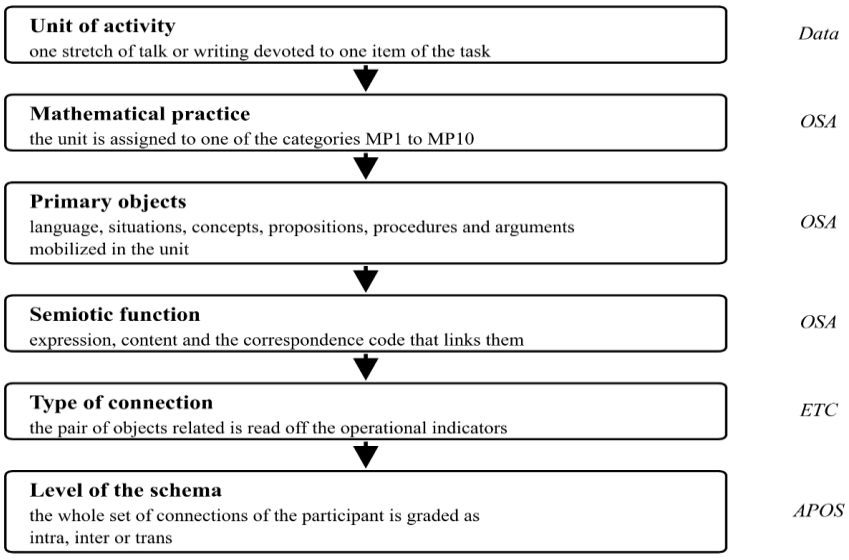



**Figure 2**. Analytical chain from the observable activity to the level of the schema

## 2.6. A Priori Analysis of the Task

Before analysing the students' productions we carried out an a priori analysis of the task, that is, an analysis of the activity a solver would have to deploy, not of what any participant actually did. It fixes the categories used later to read the data and makes the coding auditable. Everything in this subsection is theoretical and none of it is evidence about the six participants. It yields the mathematical practices MP1 to MP10, the cognitive configuration summarized in Table 3, and the genetic decomposition of Table 4.

### *System of Mathematical Practices*

The analysis of the task shows that the student's mathematical practices are organized as a progressive sequence that articulates graphical interpretations, symbolic translations, and verbal arguments, allowing the behavior of the function $f$ to be reconstructed from its derivative $f'$.

MP1. Interpretation of the sign of $f'$: the student analyzes the position of the graph of $f'$ relative to the x-axis in order to identify intervals where $f' > 0$ or $f' < 0$, establishing a first approximation to the behavior of the function $f$.

MP2. Determination of the increase and decrease of $f$: based on the sign of $f'$, the student determines the intervals where the function $f$ is increasing or decreasing, articulating graphical, symbolic, and verbal representations.

MP3. Identification of critical points: the student locates the points where $f'(x) = 0$ (intersections with the x-axis) and recognizes them as candidates for relative extrema.

MP4. Analysis of sign changes of $f'$: the student analyzes the behavior of $f'$ before and after its zeros in order to determine the existence of relative maxima and minima of $f$.

MP5. Interpretation of the variation of $f'$: the student examines whether $f'$ is increasing or decreasing on the intervals, interpreting this behavior as variation of the slope of $f$.

MP6. Determination of the concavity of $f$: based on the analysis of the variation of $f'$, the student identifies intervals where the function f is concave up or concave down.

MP7. Identification of inflection points: the student recognizes that changes in the variation of $f'$ (local maxima or minima of $f'$) correspond to possible inflection points of $f$.

MP8. Local construction of the graph of $f$ (part-whole): the student builds the graph of $f$ interval by interval, integrating the information obtained about increase, extrema, and concavity to generate coherent segments of the function.

MP9. Global integration of the graph of $f$: the student coordinates all the previous elements to construct a coherent global representation of the function $f$, ensuring consistency between its local properties and its overall behavior.

MP10. Argumentation and validation of the sketch: the student justifies their decisions through arguments that articulate relationships between $f'$ and $f$, validating the coherence of the sketch constructed.

### *Cognitive Configuration (Task: Reconstruction of f from f')*

The four items of the task structure the activity and guide the construction of the meaning of the relationship among $f$, $f'$ and $f''$, since they require in turn the intervals of monotonicity, the relative extrema, the concavity and the inflection points, and then the

assembly of all of them into one sketch. The languages that emerge in the resolution are summarized in Table 3.

**Table 3**. Emerging languages in the solution of the task

| Type of language | Examples in practice |
|---|---|
| Verbal | Increasing, decreasing, relative maximum, relative minimum, concavity, inflection point, behavior |
| Symbolic | $f, f', f'', f'(x) > 0, f'(x) < 0, f'(x) = 0, f''(x) > 0, f''(x) < 0$. |
| Graphical | Graph of $f'$, intersections with the $x$-axis, positive/negative intervals, slope of $f$, sketch of $f$. |
| Numerical | Evaluation of signs by intervals |
| Metaphorical | "goes up/goes down", "it curves upwards/downwards" |

Source: Authors' own elaboration

The definitions mobilized are that a function is increasing on an interval when its derivative is positive there and decreasing when it is negative, that a relative maximum occurs where $f'$ changes from positive to negative and a relative minimum where it changes from negative to positive, that the concavity of f depends on whether $f'$ increases or decreases, that an inflection point occurs where the concavity changes, and that the derivative is the slope of the tangent line at a point. The propositions mobilized are the conditionals that each of those definitions licenses, read in the direction that goes from $f'$ to f, together with one that no definition supplies, namely that the global behavior of f is built from the interval-by-interval analysis of $f'$.

The main procedure is to reconstruct a possible graph of f from the information supplied by $f'$. It breaks down into ten auxiliary procedures that run in the order in which the practices MP1 to MP9 were described above, from locating the zeros of $f'$ to integrating the intervals into a global graph. The arguments that justify these steps follow one pattern. A thesis is stated, taken from one of the propositions above, the sign or the variation of $f'$ on the interval under discussion is examined, and a property of f is concluded. The five arguments of the a priori analysis instantiate that pattern for increase, relative extrema, concavity, inflection points and the global construction of the graph.

### *A View from APOS Theory (Genetic Decomposition)*

**Table 4**. Genetic decomposition of the task

| Construction | Content |
|---|---|
| Actions | Identify the sign of $f'$ on each interval from its position with respect to the axis, translate that reading into $f' > 0$ or $f' < 0$ and into the words increasing and decreasing, locate the zeros of $f'$, observe whether $f'$ grows or diminishes, detect its local extrema, and sketch partial segments of $f$. |
| Processes | Interiorize and coordinate the relation between the sign of $f'$ and the monotonicity of $f$, between the zeros and sign changes of $f'$ and the relative extrema of $f$, and between the variation of $f'$ and the concavity of $f$. Coordinate the extrema of $f'$ with the changes of concavity to obtain the |

| Construction | Content |
|---|---|
| | inflection points, and integrate the interval-by-interval analysis into the global behavior of $f$. |
| Objects | Monotonicity, relative extrema, concavity, inflection points and global behavior of $f$, each encapsulated from the corresponding process. |
| Schema | A coherent system of relations among the sign of $f'$, its zeros, its variation, the monotonicity, the extrema, the concavity, the inflection points and the graph, which allows the student to read the graph of $f'$ as information about $f$, build $f$ interval by interval and integrate the local behaviors into a consistent sketch. |

Source: Authors' own elaboration

### *Operational Criteria for the Analysis*

Table 5 gives, for each type of connection, the indicator that has to be observed in a unit of activity, and Table 6 gives the criterion applied to the complete record of a participant. Between them the two tables cover the seven intra-mathematical types that this task activates, and a unit of activity can satisfy more than one of the indicators, as item b) in Table 7 anticipates.

**Table 5**. Operational indicators used to identify each type of connection

| Connection | Indicator that has to be observed in the unit of activity |
|---|---|
| Implication | The student states a conditional whose antecedent is a property of $f'$ and whose consequent is a property of $f$, and uses it to decide. A mere juxtaposition of the two properties without the inferential link does not count. |
| Reversibility | The student runs the relation in both directions, from $f$ to $f'$ and back, within the same unit. One direction alone does not count. |
| Meaning | The student says what the derivative is, not only what it does. A statement of the type the derivative is the slope of the tangent line counts, whereas naming a property such as it is positive here does not. |
| Part-whole | The student treats an interval as a part of the domain and states how it fits into the global curve, or the reverse. |
| Different representations | The same object is expressed in two registers within the same unit and the student moves between them, for instance reading a position on the graph and writing the corresponding inequality. |
| Procedural | The student chains at least three auxiliary procedures in order and justifies the order, rather than executing them in isolation. |
| Feature | The student states a characteristic or a property that describes $f$ or $f'$ itself, such as having a relative extremum or a tangent line of zero slope at that point, and not only the inference drawn from that property. |

Source: Authors' own elaboration

**Table 6**. Criteria used to assign the level of development of the schema

| Level | Criterion applied to the complete record of the participant |
|---|---|
| Intra | Elements are used in isolation. No semiotic function relates a property of $f'$ to a property of $f$, and the relation between the two graphs, when it is stated, rests on a perceptual resemblance rather than on a mathematical criterion. |
| Inter | Semiotic functions relate the two functions and at least two types of connection are established, but the coordination breaks down somewhere. Typically the first derivative is handled and the second is not, so monotonicity is right while concavity or the inflection points are not, and the parts are not assembled into a coherent whole. |
| Trans | The connections are coordinated across the four items. The student relates $f$, $f'$ and $f''$, moves in both directions between them, assembles the intervals into one curve and justifies the result, so that a change in one local decision is seen to affect the global sketch. |

Source: Authors' own elaboration

### *Semiotic Functions*

The semiotic functions (see Table 7) show that understanding begins with the interpretation of the situation and evolves through the coordination of representations, leading to connections and to a coherent global schema.

**Table 7**. Semiotic functions that organize the resolution, by item of the task

| Item | Expression | Content | Practice | Connection |
|---|---|---|---|---|
| Statement | The task as a whole | Reconstructing $f$ from $f'$ | MP1 | Meaning |
| a) Monotonicity | Position of the graph of $f'$ with respect to the axis, and the expressions $f'(x) > 0$ and $f'(x) < 0$ | Intervals of increase and decrease of $f$ | MP1 and MP2 | Different representations, implication |
| b) Extrema | Intersections of $f'$ with the axis and the change of sign around them | Relative maxima and minima of $f$ | MP3 and MP4 | Implication, feature |
| c) Concavity | Variation of $f'$, read as the sign of $f''$ | Intervals of concavity of $f$ | MP5 and MP6 | Reversibility, implication |
| d) Inflection | Extrema of $f'$ | Change of concavity of $f$ | MP7 | Reversibility, implication |
| Sketch | The set of intervals and points, and the arguments that justify them | The global graph of $f$ | MP8 to MP10 | Part-whole, procedural |

Source: Authors' own elaboration

# 3. RESULTS AND DISCUSSION

## 3.1. Results

### 3.1.1. Mathematical Connections and Local Synthesis

The mathematical connections identified in this study emerge as the result of the coordination between the mathematical practices (MP1–MP10) carried out by the students (P1–P6) and the semiotic functions that mediate the construction of meaning. The connections are not isolated relationships but visible manifestations of a deeper system that integrates primary objects (OSA) and cognitive constructions (APOS).

***Implication-type Mathematical Connections***

In particular, implication-type connections are evidenced in students P1, P2, P3, P4, and P6 when, building on the initial practices of interpreting the graph of $f'$ (MP1 and MP2), they establish relationships between the sign of the derivative and the behavior of the function. These connections are built through semiotic functions that allow the transition from the graphical representation (position of $f'$ relative to the $x$-axis) to symbolic expressions ($f'(x) > 0$, $f'(x) < 0$) and then to verbal interpretations (increase or decrease of $f$). In this way, implication is not merely a logical rule but the result of a chain of signification mediated by SFs, which from APOS begins in actions (identifying signs), is interiorized in processes (sign–monotonicity coordination), and is consolidated into objects (monotonicity). In students P1, P2, P4, and P6 this articulation appears consistently across all intervals (trans level), whereas in P3 it is only partial (inter level) (e.g., P1 and P2, Figure 3):

P1: Here (he pointed with his finger), since the derivative is positive up to here, then the function has to be increasing up to this point ($x = 1$). Then, the derivative starts to be negative, so the function starts to decrease, up to the point at $x = 2$, up to the point with $x = 3$, because it is still negative. After $x = 3$, the derivative starts to be positive, so the function $f$ starts to increase. After $x = \frac{7}{2}$, the graph of $f'$ is still positive, so the graph of $f$ starts to increase again.

P2: Here in $f'$ I have positive values, these positive values represent the increase of $f$, so from the open interval $(-\infty, 1)$ there is an increase. Then from the open interval $(3,4)$ there are positive values of $f'$, that means there is an increase, from $(4, +\infty)$ there are still positive values, so there is an increase there in $f$. The decrease of $f$ is represented by $f'$ with negative values, on $(1,2)$, the open interval, $f$ is decreasing, on $(2,3)$ it is also decreasing, because in $f'$ there are negative values, so $f$ is decreasing.

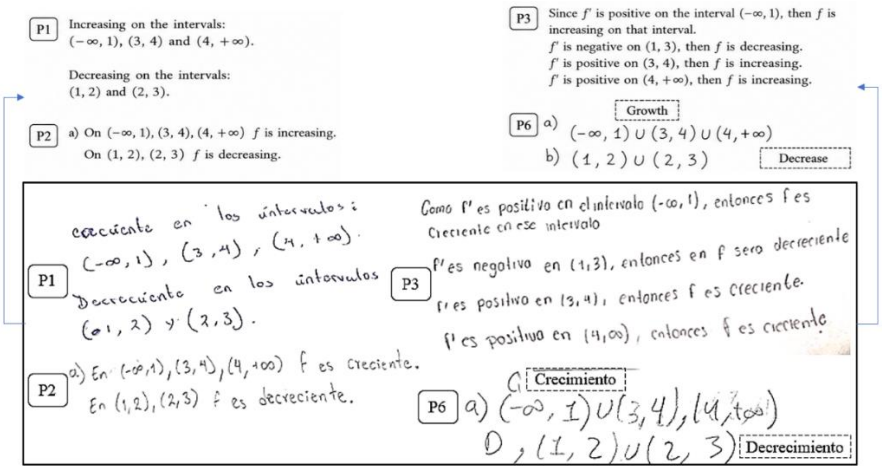



**Figure 3**. Mathematical implication connection

Likewise, these implication connections expand in the practices associated with the identification of extrema (MP3–MP4), where students coordinate the zeros of $f'$ with sign changes to determine relative maxima and minima. Here, semiotic functions play a fundamental role by relating the graphical intersection with the $x$-axis to the symbolic condition $f'(x) = 0$ and its conceptual interpretation as a critical point. This articulation enables implication to evolve toward more complex relationships, such as the determination of extrema and inflections, evidencing progress in the structuring of the schema (see transcript excerpts and Figure 4).

P2. In item b) the maximum and minimum values of $f$, this is $f'$, since the value of the slope goes from positive to negative and at $x = 1$ it equals zero, that means that $f$ goes from increasing to decreasing and at $x = 1$ there is a relative extremum, and since it goes from increasing to decreasing, it is a maximum. Then, from $x = \frac{5}{2}$ to $x = \frac{7}{2}$, $f'$ goes from negative to positive, that means it goes from decreasing to increasing, and at $x = 3$, since it equals zero, there is a relative extremum, and since I know that $f$ goes from decreasing to increasing, there is a relative minimum. Where the signs of the slopes change, that is where we have the relative extrema.

P4. The maximum points of $f$ would be where the derivative equals zero, so the original $f$ has a maximum point at $x = 1$, and a minimum at $x = 3$.

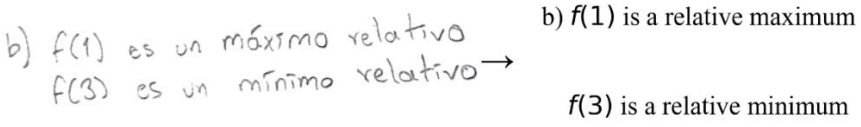



**Figure 4**. Evidence of the implication connection for relative extrema

Another implication connection was observed when students considered criteria for building the graph of $f$, and in turn it was found that P1, P2, P3, P4, and P6 recognized that $f$ is concave down on the intervals $\left(-\infty, \frac{3}{2}\right)$, $\left(2, \frac{5}{2}\right)$, and $\left(\frac{7}{2}, 4\right)$, while considering that $f$ is

concave up (convex) on the intervals $\left(\frac{3}{2}, 2\right)$, $\left(\frac{5}{2}, \frac{7}{2}\right)$, and $(4, +\infty)$. See below for the transcript excerpt of P2, their written work, and Figure 5.

P2. Where $f'$ has its relative extrema, where its slope equals zero, for example, at $x = \frac{3}{2}$, $x = 2$, $x = \frac{5}{2}$, $x = \frac{7}{2}$, and at $x = 4$, that is where $f'$ has relative extrema, that indicates that these are the inflection points of $f$, that is, when the second derivative of $f$ equals zero, then there will be inflection points in $f$. In c) we look for the concavities. Since I have the inflection points, I already know where it changes from one concavity to the other. If in $f\left(\frac{3}{2}\right)$ I have an inflection point, and I know that out here it goes from increasing to decreasing, then on the interval $\left(-\infty, \frac{3}{2}\right)$, $f$ is concave down. From $\left(\frac{3}{2}, 2\right)$, I already said that it is concave on this interval $\left(-\infty, \frac{3}{2}\right)$. If at $x = \frac{3}{2}$ there is an inflection point, that means that the concavity changes, so it must be concave up. There is another interval where there is another concavity, between $x = 2$ and $x = \frac{5}{2}$. Since there must be a change of concavity, and we already said that the previous one is concave up, then this one will be concave down. From $x = \frac{5}{2}$ to $x = \frac{7}{2}$ it is concave up, and this can be seen because it goes from negative to positive values in $f'$ and there is a point at which it equals zero, that is, there is a relative extremum. In this case, since it was decreasing and then increasing, it has a relative minimum, so it is concave up. From there, from $x = \frac{7}{2}$ to $x = 4$ there is another concavity that must be down because it is right at a change, and on $(4, +\infty)$ it is concave up.

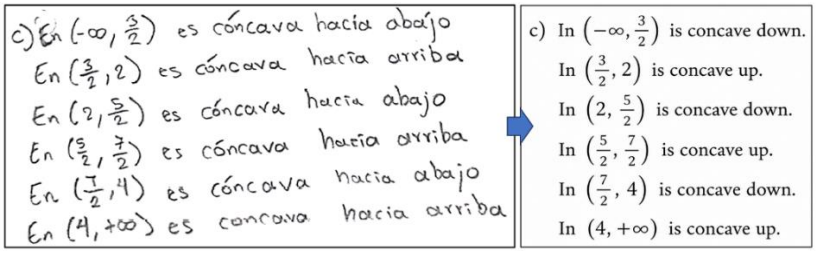



**Figure 5**. P2's written work on the intervals of concavity

Once the intervals of concavity were identified, the students established that a change of concavity implies an inflection point. Likewise, P1, P2, P4, and P6 expressed the relationship among $f$, $f'$, and $f''$, recognizing that at the point where the graph of $f'$ has a maximum or a minimum, at that point the second derivative $f''$ equals zero and, therefore, $f$ has an inflection point. As an example, we present a transcript excerpt from P1 and the written work of P2 and P6, together with Figure 6.

P1. At $x = \frac{7}{2}$ there is a maximum for $f'$, so it would be an inflection point for $f$, at this point here, so it changes concavity. Then, here ($x = 4$) as well, it would be an inflection point, actually it does not cross the $x$-axis, it only touches it, so $f$ changes, it goes up and comes back down.

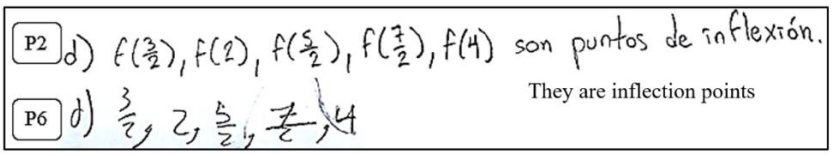



**Figure 6**. Written work of P2 and P6 on the inflection points of $f$

### *Reversibility Mathematical Connections*

Reversibility connections are evidenced in P1, P2, P4 and P6 in items c and d, where the direction of the reading has to be inverted. These students move from the graph of $f'$ to a property of $f$ and then back from that property to the variation of $f'$, in order to decide the concavity and to locate the inflection points. The excerpts presented above record that double movement, since the extrema of $f'$ are read as inflection points of $f$ and those points are then used to fix the intervals of concavity. In P3 the second derivative never enters the reasoning and in P5 the two graphs are related by a perceptual criterion, so in neither case is the relation run in both directions.

### *Mathematical Connections of Meaning*

Meaning connections are evidenced in P1 and P2 when they interpret the derivative as the slope of the tangent line. This connection is built through semiotic functions that articulate the graphical representation, the verbal meaning of slope, and the symbolic expression $f'$. It is especially activated in MP1 and MP5, when students use the slope to justify the increase, the decrease, and the shape of the graph. From the networking perspective, it constitutes the conceptual basis of implication connections. From APOS, it shows the transition from the action—recognizing the slope—to the object—derivative as a rate of change—although with different degrees of consolidation. Nevertheless, this meaning is not always generalized, which limits other mathematical connections.

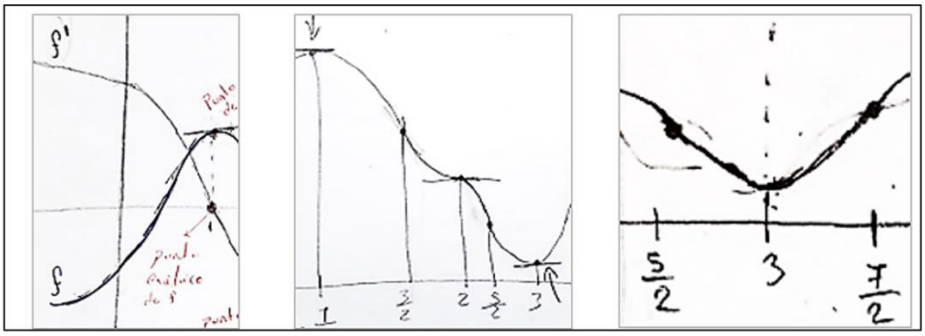
I. When you say $f'$, what are you actually referring to?

P1. To the slope of the tangent lines to $f$, here they would be positive, until they get close to this critical point of $f$ and here the slope is zero. Then the graph starts to be negative and the slopes of the graph for $f$ are now negative.

P1 and P2 therefore establish a meaning connection that draws on a partial meaning of the derivative.

### *Part-whole Mathematical Connections*

Part-whole connections are evidenced in the practices of interval-based graph construction (MP8 and MP9), where P1, P2, P3, P4, and P6 organize the solution according to the local behavior of the function. The semiotic functions allow each interval of the graph of $f'$ to be associated with specific behaviors of $f$, building segments that are then integrated into a global representation. This connection is linked to APOS integration processes, by coordinating intervals, extrema, and concavity to form the complete graph as a coherent schema (see Figure 7).




**P1:** on the interval $(-\infty, 1)$ (part), $f' > 0$, therefore $f$ on $(-\infty, 1)$ is increasing.

**P6:** on the intervals $(1, 2)$ and $(2, 3)$ (part), $f' < 0$, therefore $f$ on those intervals is decreasing.

**P2:** on the interval $\left(\frac{5}{2}, \frac{7}{2}\right)$ (part), $f$ is concave up (convex).

**Figure 7**. Part-whole mathematical connection

Figure 7 shows the inclusion relationship: the students sketched $f$ by intervals according to the behavior of $f'$.

### *Mathematical Connections Between Different Representations*

Connections between different representations are evidenced in the practices MP2, MP4, MP6, and MP10, where students coordinate graphical, symbolic, and verbal registers. These connections, mediated by semiotic functions, allow the translation of information to interpret the graph of $f'$, to express symbolic conditions, and to argue results. In P1, P2, P4, and P6 coordination is consistent and fosters the global construction of the function, placing them at the trans level, whereas in P3 it is partial and in P5 it is not evidenced. Likewise, Figures 8 and 9 evidence verbal, written, graphical, and symbolic connections.

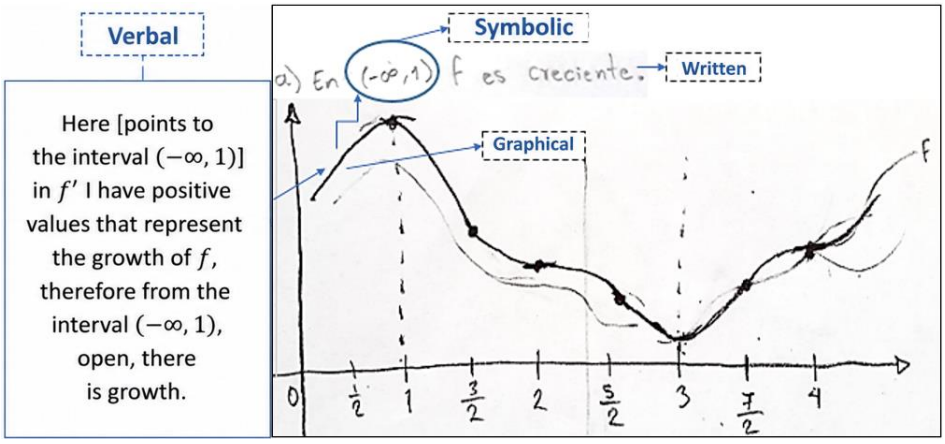



**Figure 8**. Mathematical connections between different representations

In Figure 9, P1 built both graphs ($f$ and $f'$) to show their relationships and marked the maximum, minimum, and inflection points.

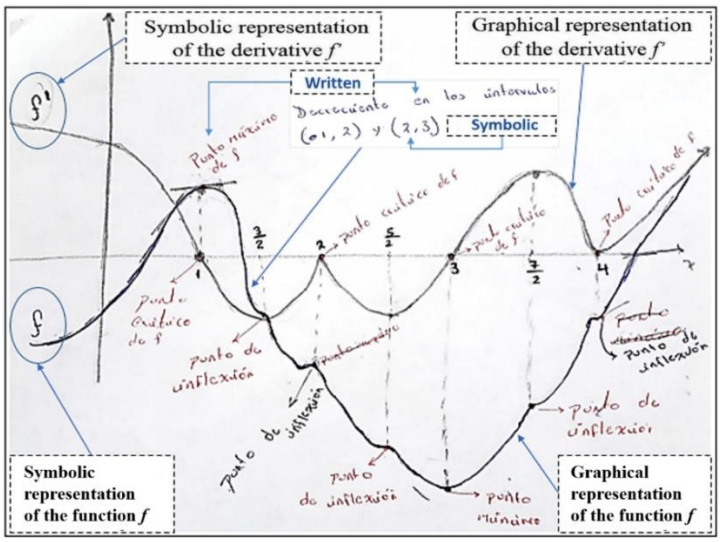



**Figure 9**. Sketch of the graph of $f$ considering the implication-type mathematical connections, or relationships among $f$, $f'$, and $f''$

### *Procedural Mathematical Connections*

Procedural connections integrate the previous ones and are evidenced in the sequence of practices used to solve the task. P1, P2, P4, and P6 developed procedures that articulate the analysis of signs, the identification of extrema, the study of concavity, and the global verification of the graph. These connections are supported by multiple semiotic functions and by the coordination of practices, which from APOS is interpreted as the consolidation of the schema (see the case of P2 in Figure 10).

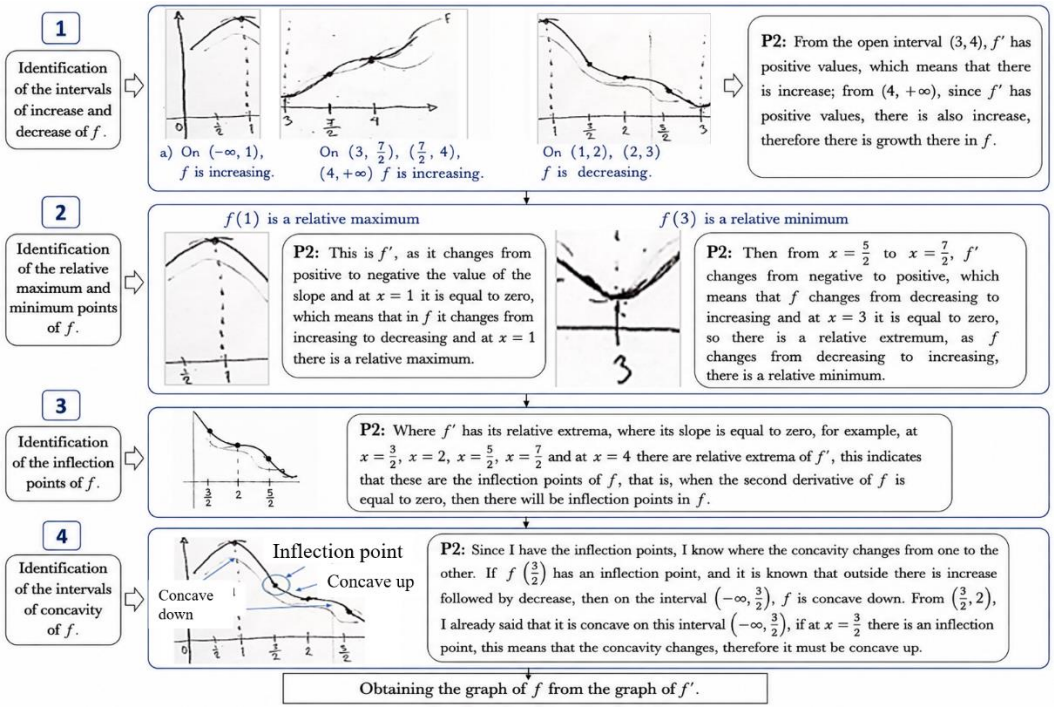



**Figure 10**. Procedural connection made by P2 to build the graph of $f$

***Feature Mathematical Connections***

This connection became evident when students identified the characteristics, properties and essential elements of the two functions, $f$ and $f'$. P1 and P2 noted that the function is increasing or decreasing, that it has a maximum or a minimum and that it changes concavity at a certain point, and they observed that at $x = 3$ the graph of the derivative intersects the $x$-axis while the function itself reaches a minimum (see Figure 4). At the maxima and the minima, they also drew tangent lines of zero slope, parallel to the $x$-axis.

### 3.1.2. Students' Understanding Through Mathematical Connections

Students' understanding is configured as a network of connections that emerges from the coordination among mathematical practices, primary objects, and semiotic functions that make it possible to construct meaning while solving the task.

Students P1, P2, P4, and P6 show a trans level of understanding, characterized by the consistent articulation of implication, different-representations, reversibility, part-whole and procedural connections across the mathematical practices, with connections of meaning and of feature in P1 and P2. From the first practices (MP1–MP2), they interpret the sign of $f'$ through semiotic functions that link graphical representation, symbolic expressions, and verbal meanings. Later, in more advanced practices (MP3–MP7), they coordinate these relationships to identify relative extrema, intervals of concavity, and inflection points, establishing more complex implication connections that involve the relationships among $f$, $f'$, and $f''$. This progression evidences the transition from initial actions to coordinated processes and the consolidation of mathematical objects such as monotonicity, concavity, and inflection, in APOS terms.

In the global construction (MP8–MP10) they integrate the information by intervals through part-whole connections, achieving a coherent representation of $f$. Here semiotic functions play a central role, since they allow graphical, symbolic, and verbal representations to be translated and coordinated, facilitating the construction of the schema. Consequently, these students reach a trans level of understanding, in that they not only establish local relationships but also manage to integrate them into a coherent global structure, in accordance with the characterizations of Sánchez-Matamoros (2004) and Borji et al. (2018).

Student P3, in turn, is placed at an inter level of understanding because, although he establishes relevant connections between $f$ and $f'$, these are not articulated in a complete way. In particular, P3 manages to activate semiotic functions associated with the sign of the derivative and the increase of the function (MP1–MP2), as well as to identify some relative extrema (MP3–MP4). He shows difficulties, however, in the coordination of representations and in building implication connections related to concavity and inflection points. This is evidenced in the partial identification of those points, recognizing some correct values (such as $x = \frac{3}{2}$ and $x = \frac{7}{2}$), but omitting others (such as $x = 2$ and $x = 4$), which limits the global integration of the behavior of the function. In networking terms, this indicates that the activated semiotic functions do not manage to be fully articulated into a coherent network of connections, remaining restricted to local relationships (see Figure 11 and Table 8).

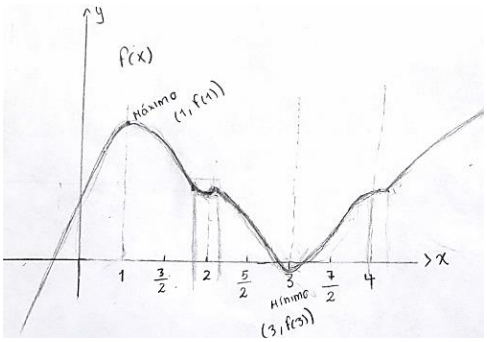



**Figure 11**. Graphical representation of $f$ from $f'$ produced by P3

In contrast, student P5 is placed at an intra level of understanding, due to the absence of meaningful mathematical connections. In this case, there are no coherent mathematical practices that establish relationships between $f$ and $f'$, nor semiotic functions that articulate the representations involved. The interpretation of the graph of $f$ as a "reflection" of $f'$ reveals a fragmented understanding, in which the meaning of the derivative as the slope of the tangent line is not recognized, nor is its relationship with the behavior of the function. This lack of coordination among practices, objects, and representations prevents the construction of implication, meaning, or part-whole connections, which translates into an inability to build the schema required to solve the task, as shown in the following audio excerpt and in Figure 12.

P5. A relationship between the graph of $f'$ and $f$, you could say it is like its reflection, where it is concave down for $f$, for $f'$ it is concave up, you could say it is like an inverse.

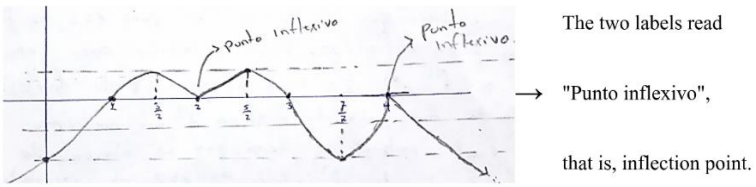



**Figure 12**. Sketch of the graph of $f$ produced by P5

Table 8 summarizes, for each participant, the connections identified in the corpus and the level of development of the schema assigned.

**Table 8**. Connections established by each participant and level assigned

| | P1 | P2 | P3 | P4 | P5 | P6 |
|---|---|---|---|---|---|---|
| Implication | Yes | Yes | Yes | Yes | Not evidenced | Yes |
| Reversibility | Yes | Yes | Not evidenced | Yes | Not evidenced | Yes |
| Meaning | Yes | Yes | Not evidenced | Not evidenced | Not evidenced | Not evidenced |
| Part-whole | Yes | Yes | Yes | Yes | Not evidenced | Yes |
| Different representations | Yes | Yes | Partial | Yes | Not evidenced | Yes |
| Procedural | Yes | Yes | Not evidenced | Yes | Not evidenced | Yes |

| | **P1** | **P2** | **P3** | **P4** | **P5** | **P6** |
|---|---|---|---|---|---|---|
| Feature | Yes | Yes | Not evidenced | Not evidenced | Not evidenced | Not evidenced |
| Level assigned | Trans | Trans | Inter | Trans | Intra | Trans |

Source: Authors' own elaboration

### 3.2. Discussion

What separates the three levels in this group is not how many connections a student establishes but whether the connections hold together when one of them is put under pressure. The four participants placed at trans moved between $f$, $f'$ and $f''$ in both directions, so that a decision about concavity could be checked against a decision about monotonicity. P3 established implication and part-whole connections as well, and read the sign of $f'$ correctly, but the second derivative never entered the reasoning, so the concavity was decided by the shape of the drawing rather than by the variation of $f'$. That is the signature of the inter level, a coordination that works locally and breaks at the point where two levels of derivation have to be held together. P5 did not establish a semiotic function between the two functions at all. The correspondence code he used was perceptual, the graph of $f$ as a mirror image of the graph of $f'$, which explains why the answer is wrong in a systematic rather than a scattered way.

The implication connection and the part-whole connection were the most widespread in this group, each present in five of the six records, which is consistent with what García-García and Dolores-Flores (2018, 2021) report for pre-university students solving calculus tasks. The present data add a qualification to that finding. In those studies the implication connection appears when students move from an algebraic condition to a property of the function, whereas here it had to be built from a graph with no formula available. Five of the six participants established it, yet only four ran the relation between $f'$ and $f$ in both directions, and it is that difference, rather than the presence of the connection, that separated the levels. Counting connections does not capture the distinction, whereas the semiotic function makes it visible, because it shows which correspondence code the student actually used.

The reversibility connection behaved as Rodríguez-Nieto et al. (2023) and Rodríguez-Nieto and Font Moll (2025) would predict, in that the students who ran the relation between $f$ and $f'$ in both directions were the ones who produced a coherent global sketch. The interesting case is P5, whose reading of one graph as the mirror image of the other matches the difficulty Rodríguez-Nieto et al. (2025) describe. The analysis here locates where that reading comes from. It is not a failure to remember the criterion but the use of a perceptual correspondence code in place of a mathematical one, and because the code is applied consistently the resulting graph is wrong in an orderly way. Fuentealba, Badillo and Sánchez-Matamoros (2019) report a comparable pattern when students work with the subschemas of the derivative, and the present data suggest that the semiotic function is what distinguishes a wrong answer produced by an invalid criterion from a wrong answer produced by carelessness.

The contribution of the ETC–APOS–OSA networking can now be stated in terms of what the three frameworks supply one another. Taken separately, APOS would have placed the six participants on the intra, inter and trans scale without saying which observable fragment of their activity justifies the placement. The ETC would have listed the connections without a criterion for deciding, in a contested case, whether a given utterance is a connection of meaning

or of feature. The OSA would have described the activity in detail but would not have graded it. In the analysis reported here the semiotic function supplies the missing criterion for the ETC, because the pair of objects it relates and the correspondence code that links them decide which connection is at stake, and the APOS level supplies the grading, because it names how far the connections identified in that way have been coordinated. This is what previous APOS studies of the derivative have not addressed. Borji et al. (2018) articulate APOS with the OSA and describe the objects a student mobilizes, yet the step from those objects to a level of the schema remains a judgement of the analyst rather than a documented inference. Adding the ETC turns that step into a chain that can be audited, from practice to object, from object to semiotic function, from semiotic function to connection and from the set of connections to the level.

Three consequences for teaching follow from this, within the limits set out below. The first concerns task design. A graphical task elicits reversibility only if the algebraic expression is withheld, because as soon as a formula is available the student can differentiate and the reverse reading is never required. The second concerns what a teacher should listen for. A correct graphical answer can rest on an invalid criterion, as P5 shows, so asking the student to say why a stretch of the curve goes up is more informative than checking whether it goes up. The third concerns the order of the items. Because the difficulty concentrated where the second derivative had to be read off the variation of the first, a sequence that separates that step from the first derivative criterion, and makes it explicit, is likely to be more useful than one that presents the four questions as variants of a single procedure.

The study has four limitations that bound these claims. It rests on six students at one Mexican university who volunteered, so the group is self-selected and the results describe it rather than any wider population. It rests on one task, taken from Spivak (1992), so what is reported is how these connections operate in the reconstruction of a function from the graph of its derivative, and the model is not shown to transfer to other topics or to other graphical contexts. The assignment of connections and levels is an interpretation built on the researchers' reading of the transcripts, and although it was coded independently by two authors and the criteria are stated in Tables 5 and 6 so that the inference can be followed and contested, it remains interpretative. And the levels were assigned at a single moment, so nothing here speaks to how a student moves from one level to another over time. A study with more participants, several tasks and two points in time would be needed to test whether the chain described here holds beyond this case.

## 4. CONCLUSION

This study asked which mathematical connections university students establish when they reconstruct the graph of a function from the graph of its derivative, through which semiotic functions those connections are built, and how their coordination relates to the levels of development of the derivative schema. In this group of six, four of the six participants reached a trans level of schema development by coordinating implication, representation, reversibility, procedural and part-whole connections, two of them also mobilizing connections of meaning and of feature, one participant remained at an inter level because the second derivative never entered his reasoning, and one remained at an intra level because he related

the two graphs by a perceptual criterion rather than a mathematical one. The connections were built through semiotic functions whose correspondence code is what decides, in each case, which connection is at stake. The contribution of articulating the ETC, APOS and the OSA is that it makes the inference from observable activity to cognitive level auditable, since the semiotic function supplies the criterion that identifies the connection and the level names how far those connections have been coordinated. Because the evidence comes from one task and one small group, the model is offered as a local one. Testing it with more participants, more tasks and more than one point in time is the natural continuation, as is examining whether a teaching sequence that makes the passage between the first and the second derivative explicit moves students who stall at the inter level.

## Acknowledgments

The authors would like to extend their sincere gratitude to the National Research and Development Agency (ANID) for funding this work through Fondecyt Regular Project No. 1241700.

## Declarations

Author Contribution : CAR-N: Investigation, Methodology, Project administration, Visualization, Writing - original draft, and Writing - review & editing; CF: Investigation, Methodology, Writing - original draft, and Writing - review & editing; VFM: Formal analysis, Investigation, Methodology, Resources, and Writing - review & editing; FMR-V: Resources, Supervision, Validation, and Writing - review & editing; AC: Funding acquisition, Resources, Supervision, Validation, and Writing - review & editing.

Funding Statement : This work was funded by the National Research and Development Agency (ANID) through Fondecyt Regular Project No. 1241700.

Conflict of Interest : The authors declare no conflict of interest.

Additional Information : Additional information is available for this paper.

## REFERENCES

Ariza, Á., & Llinares Ciscar, S. (2009). Sobre la aplicación y uso del concepto de derivada en el estudio de conceptos económicos en estudiantes de bachillerato y universidad [On the application and use of the concept of the derivative in the study of economic concepts among high school and university students]. *Enseñanza de Las Ciencias. Revista de Investigación y Experiencias Didácticas*, *27*(1), 121–136. https://doi.org/10.5565/rev/ensciencias.3667

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Roa Fuentes, S., Trigueros, M., & Weller, K. (2014). *APOS theory: A framework for research and curriculum development in mathematics education*. Springer. https://doi.org/10.1007/978-1-4614-7966-6

Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In J. Kaput, A. H. Schoenfeld, & E. Dubinsky (Eds.), *Research in collegiate*

*mathematics education II* (Vol. 6, pp. 1–32). Conference Board of the Mathematical Sciences (CBMS).

Badillo, E., Azcárate, C., & Font, V. (2011). Análisis de los niveles de comprensión de los objetos f'(a) y f'(x) de profesores de matemáticas [Analysis of mathematics teachers' levels of understanding of the objects f'(a) and f'(x)]. *Enseñanza de Las Ciencias. Revista de Investigación y Experiencias Didácticas*, *29*(2), 191–206. https://doi.org/10.5565/rev/ec/v29n2.546

Berry, J. S., & Nyman, M. A. (2003). Promoting students' graphical understanding of the calculus. *The Journal of Mathematical Behavior*, *22*(4), 479–495. https://doi.org/10.1016/j.jmathb.2003.09.006

Bikner-Ahsbahs, A., & Johnson, H. L. (2026). Foundation, form, and function: the "3F" framework to dialogue within and across theories in mathematics education research. *Frontiers in Education*, *11*, 1773672. https://doi.org/10.3389/feduc.2026.1773672

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories—An approach for exploiting the diversity of theoretical approaches. In B. Sriraman & L. English (Eds.), *Theories of mathematics education* (pp. 483–506). https://doi.org/10.1007/978-3-642-00742-2_46

Borji, V., Font, V., Alamolhodaei, H., & Sánchez, A. (2018). Application of the complementarities of two theories, APOS and OSA, for the analysis of the university students' understanding on the graph of the function and its derivative. *Eurasia Journal of Mathematics, Science and Technology Education*, *14*(6), 2301–2315. https://doi.org/10.29333/ejmste/89514

Businskas, A. M. (2008). *Conversations about connections: How secondary mathematics teachers conceptualize and contend with mathematical connections*. Tesis doctoral. Simon Fraser University.

Cai, J., & Ding, M. (2017). On mathematical understanding: perspectives of experienced Chinese mathematics teachers. *Journal of Mathematics Teacher Education*, *20*(1), 5–29. https://doi.org/10.1007/s10857-015-9325-8

Cohen, L., Manion, L., & Morrison, K. (2013). *Research methods in education* (7th ed.). Routledge.

Dolores-Flores, C., & García-García, J. (2017). Conexiones intramatemáticas y extramatemáticas que se producen al resolver problemas de cálculo en contexto: Un estudio de casos en el nivel superior [Intra-mathematical and extra-mathematical connections that arise when solving calculus problems in context: A case study at the higher education level]. *Bolema: Boletim de Educação Matemática*, *31*(57), 158–180. https://doi.org/10.1590/1980-4415v31n57a08

Dubinsky, E., & Mcdonald, M. A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In D. Holton, M. Artigue, U. Kirchgräber, J. Hillel, M. Niss, & A. Schoenfeld (Eds.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 275–282). Springer. https://doi.org/10.1007/0-306-47231-7_25

Eli, J. A., Mohr-Schroeder, M. J., & Lee, C. W. (2013). Mathematical connections and their relationship to mathematics knowledge for teaching geometry. *School Science and Mathematics*, *113*(3), 120–134. https://doi.org/10.1111/ssm.12009

Font Moll, V., Trigueros, M., Badillo, E., & Rubio, N. (2016). Mathematical objects through the lens of two different theoretical perspectives: APOS and OSA. *Educational Studies in Mathematics*, *91*(1), 107–122. https://doi.org/10.1007/s10649-015-9639-6

Font, V., Godino, J. D., & Gallardo, J. (2013). The emergence of objects from mathematical practices. *Educational Studies in Mathematics*, *82*(1), 97–124. https://doi.org/10.1007/s10649-012-9411-0

Fuentealba, C., Badillo, E., & Sánchez-Matamoros, G. (2018). Puntos de no-derivabilidad de una función y su importancia en la comprensión del concepto de derivada [Points of non-differentiability of a function and their importance in understanding the concept of the derivative]. *Educação e Pesquisa*, *44*, e181974. https://doi.org/10.1590/s1678-4634201844181974

Fuentealba, C., Badillo, E., & Sánchez-Matamoros, G. (2019). Identificación y caracterización de los subniveles de desarrollo del esquema de derivada [Identification and characterization of the developmental sub-levels of the derivative schema]. *Enseñanza de Las Ciencias. Revista de Investigación y Experiencias Didácticas*, *37*(2), 63–84. https://doi.org/10.5565/rev/ensciencias.2518

Fuentealba, C., Badillo, E., Sánchez-Matamoros, G., & Cárcamo, A. (2019). The understanding of the derivative concept in higher education. *Eurasia Journal of Mathematics, Science and Technology Education*, *15*(2), em1662. https://doi.org/10.29333/ejmste/100640

Fuentealba, C., Sánchez-Matamoros, G., & Badillo, E. (2015). Análisis de tareas que pueden promover el desarrollo de la comprensión de la derivada [Analysis of tasks that can promote the development of an understanding of the derivative]. *Uno: Revista de didáctica de las matematicas*, *70*, 72–77.

Galindo-Illanes, M. K., Chamorro-Manríquez, D., Breda, A., & Sala-Sebastià, G. (2025). Instructional process for the construction of the derivative function: Modelling and simulation in GeoGebra from the ontosemiotic approach. *Journal on Mathematics Education*, *16*(3), 1001–1022. https://doi.org/10.22342/jme.v16i3.pp1001-1022

García-García, J., & Dolores-Flores, C. (2018). Intra-mathematical connections made by high school students in performing calculus tasks. *International Journal of Mathematical Education in Science and Technology*, *49*(2), 227–252. https://doi.org/10.1080/0020739x.2017.1355994

García-García, J., & Dolores-Flores, C. (2021). Exploring pre-university students' mathematical connections when solving calculus application problems. *International Journal of Mathematical Education in Science and Technology*, *52*(6), 912–936. https://doi.org/10.1080/0020739x.2020.1729429

García, M., Llinares, S., & Sánchez-Matamoros, G. (2011). Characterizing thematized derivative schema by the underlying emergent structures. *International Journal of Science and Mathematics Education*, *9*(5), 1023–1045. https://doi.org/10.1007/s10763-010-9227-2

Godino, J. D., Batanero, C., & Font, V. (2019). The onto-semiotic approach. *For the learning of Mathematics*, *39*(1), 38–43.

Guo, P.-C., & Yang, K.-L. (2026). Identifying the essential views and external contexts: the presentations and the arrangement of the interpretation of derivatives in textbooks for

different student groups. *Mathematical thinking and learning*, 1–23. https://doi.org/10.1080/10986065.2026.2663783

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 65–97). Macmillan.

Lastari, K., P. A., Agustika, G. N. S., & Sutyaningsih, A. A. D. (2025). APOS theory-based learning on the procedural fluency of fourth-grade elementary school students. *Indonesian Journal of Instruction*, *6*(2), 249–261. https://doi.org/10.23887/iji.v6i2.93659

Ledezma, C., Rodríguez-Nieto, C. A., & Font, V. (2024). The role played by extra-mathematical connections in the modelling process. *Avances de investigación en Educación Matemática*, *25*, 81–103. https://doi.org/10.35763/aiem25.6363

Matindike, F. (2025). Deepening learners' conceptualization of functions using autograph technology: An APOS theory perspective. *Asian Journal for Mathematics Education*, *4*(4), 528–552. https://doi.org/10.1177/27527263251375522

Nahari, H., Mohammadi Matin, A., & Taghavi, Z. (2025). Critical analysis of the APOS theory: A framework for curriculum development in mathematics education. *Research in Mathematics Education*, *5*(2), 49–60.

NCTM. (2000). *Principles and standards for school mathematics*. National Council of Teachers of Mathematics.

Pino-Fan, L. R., Godino, J. D., & Font, V. (2018). Assessing key epistemic features of didactic-mathematical knowledge of prospective teachers: the case of the derivative. *Journal of Mathematics Teacher Education*, *21*(1), 63–94. https://doi.org/10.1007/s10857-016-9349-8

Pino-Fan, L. R., Guzmán, I., Font, V., & Duval, R. (2017). Analysis of the underlying cognitive activity in the resolution of a task on derivability of the absolute-value function: two theoretical perspectives. *PNA. Revista de Investigación en Didáctica de la Matemática*, *11*(2), 97–124. https://doi.org/10.30827/pna.v11i2.6076

Qu, S. Q., & Dumay, J. (2011). The qualitative research interview. *Qualitative Research in Accounting & Management*, *8*(3), 238–264. https://doi.org/10.1108/11766091111162070

Ramadiana, A., Riyadi, R., Indriati, D., & Nurhasanah, F. (2026). APOS theory and relational understanding: A systematic literature review. *Mandalika Mathematics and Educations Journal*, *8*(1), 397–412. https://doi.org/10.29303/jm.v8i1.11155

Rodríguez-Nieto, C. A., Cabrales González, H. A., Arenas-Peñaloza, J., Schnorr, C. E., & Font Moll, V. (2024). Onto-semiotic analysis of Colombian engineering students' mathematical connections to problems-solving on vectors: A contribution to the natural and exact sciences. *Eurasia Journal of Mathematics, Science and Technology Education*, *20*(5), em2438. https://doi.org/10.29333/ejmste/14450

Rodríguez-Nieto, C. A., & Font Moll, V. (2025). Mathematical connections promoted in multivariable calculus' classes and in problems-solving about vectors, partial and directional derivatives, and applications. *Eurasia Journal of Mathematics, Science and Technology Education*, *21*(4), em2619. https://doi.org/10.29333/ejmste/16187

Rodríguez-Nieto, C. A., Font Moll, V., Borji, V., & Rodríguez-Vásquez, F. M. (2022). Mathematical connections from a networking of theories between extended theory of

mathematical connections and onto-semiotic approach. *International Journal of Mathematical Education in Science and Technology*, *53*(9), 2364–2390. https://doi.org/10.1080/0020739x.2021.1875071

Rodríguez-Nieto, C. A., Rodríguez-Vásquez, F. M., & Font Moll, V. (2022). A new view about connections: the mathematical connections established by a teacher when teaching the derivative. *International Journal of Mathematical Education in Science and Technology*, *53*(6), 1231–1256. https://doi.org/10.1080/0020739x.2020.1799254

Rodríguez-Nieto, C. A., Rodríguez-Vásquez, F. M., & Font Moll, V. (2023). Combined use of the extended theory of connections and the onto-semiotic approach to analyze mathematical connections by relating the graphs of f and f'. *Educational Studies in Mathematics*, *114*(1), 63–88. https://doi.org/10.1007/s10649-023-10246-9

Rodríguez-Nieto, C. A., Rodríguez-Vásquez, F. M., Font Moll, V., Sudirman, S., & Cantillo-Rudas, B. M. (2025). Engineering students' mathematical understanding based on the quality of mathematical connections activated to solve tasks about function's graph and its derivative. *Educational Process International Journal*, *17*(1), e2025394. https://doi.org/10.22521/edupij.2025.17.394

Romero Osorio, L. A., Aldana Bermúdez, E., & Gutiérrez Jiménez, J. T. (2026). Encapsulation of the defined integral mathematical object of university students: a view from APOS theory. *Cogent Education*, *13*(1), 2588533. https://doi.org/10.1080/2331186x.2025.2588533

Sánchez-Matamoros, G. (2004). *Análisis de la comprensión de los alumnos de bachillerato y primer curso de la universidad sobre la noción matemática de derivada [Analysis of the understanding of the mathematical concept of the derivative among high school and first-year university students]*. Tesis doctoral. Universidad de Sevilla.

Sánchez-Matamoros, G., Fernández, C., & Llinares, S. (2015). Developing pre-service teachers' noticing of students' understanding of the derivative concept. *International Journal of Science and Mathematics Education*, *13*(6), 1305–1329. https://doi.org/10.1007/s10763-014-9544-y

Scheiner, T. (2026). Whose frameworks travel? Towards epistemological reciprocity in mathematics teacher education research. *Journal of Mathematics Teacher Education*, *29*(4), 761–771. https://doi.org/10.1007/s10857-026-09756-z

Spivak, M. (1992). *Calculus* (2nd ed.). Publish or Perish.

Syaiful, S., Mukminin, A., & Puspayanti, P. (2025). Using APOS theory in learning basic mathematics to promote students' mathematical critical thinking skills. *Discover Education*, *4*(1), 443. https://doi.org/10.1007/s44217-025-00863-2

Vargas González, M. F., Fernández-Plaza, J. A., & Ruiz-Hidalgo, J. F. (2020). Análisis de los argumentos dados por docentes en formación a una tarea sobre derivadas [Analysis of the arguments provided by pre-service teachers for a task on derivatives]. *PNA. Revista de Investigación en Didáctica de la Matemática*, *14*(3), 173–203. https://doi.org/10.30827/pna.v14i3.12229

Voskoglou, M. G. (2017). Application of grey numbers to assessment of the understanding the graphical representation of the derivative. *American Journal of Educational Research*, *5*(11), 1167–1171. https://doi.org/10.12691/education-5-11-9